ECE 202 - Linear Circuit Analysis II

Purdue University, Spring 2009

Homework Set 7 Solutions

Some Important Derivations

Here we will derive some useful expressions for equivalent inductances of circuits containing coupled inductances in various configurations. Consider the circuit shown below.

We can write the following KVL equations for the given circuit following the

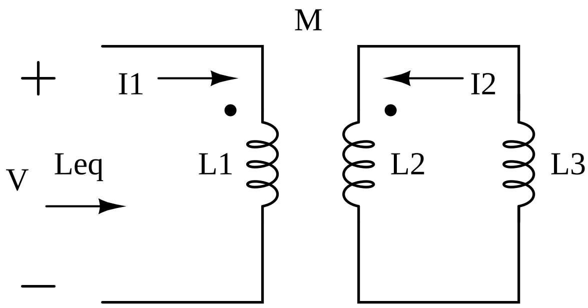

Figure 1: Configuration 1

dot convention.

$$
\begin{aligned}
V - L_1 I_1 s - M(I_2)s &= 0 \\
-L_2 I_2 s - M(I_1)s - L_3 I_2 s &= 0 \\
\Rightarrow I_2 &= -\frac{M}{L_2 + L_3} I_1 \\
\Rightarrow L_{eq} &= \frac{V}{I_1 s} \\
&= L_1 - \frac{M^2}{L_2 + L_3}
\end{aligned}
$$

Now consider the following circuit on the next page.

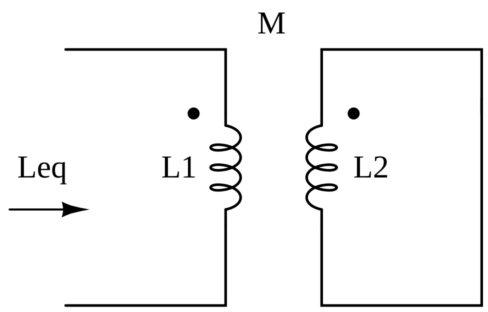

Figure 2: Configuration 2

Note that for this configuration L_3 is not present. Thus by setting $L_3 = 0$ in the previous configuration we get,

$$L_{eq} = L_1 - \frac{M^2}{L_2}$$

Now consider a third configuration.
Here we can write using our previous derivations,

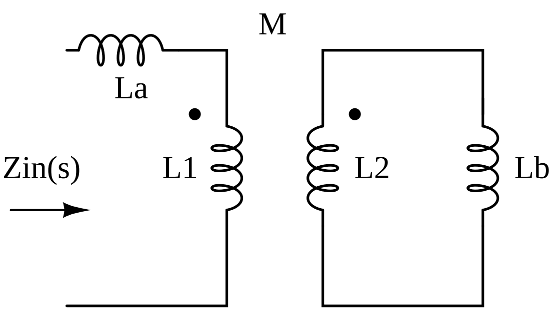

Figure 3: Configuration 3

$$Z_{in}(s) = s[L_a + L_1 - M^2/(L_2 + L_b)]$$

Exercise: Prove that the above derivations are independent of the dot convention.

Solution 61

(a)

Let V_1 and V_2 be the voltages at the left and right terminals. Writing KVL equation and following the dot convention

$$\begin{aligned}
V_1 - L_1 Is - L_2 Is - IR - \frac{I}{Cs} - M(-Is) + M(Is) &= V_2 \\
\Rightarrow Z(s) &= \frac{V_1 - V_2}{I} \\
&= (L_1 + L_2 - 2M)s + R + \frac{1}{Cs} \\
\omega_r &= \frac{1}{\sqrt{(L_1 + L_2 - 2M)C}} \\
&= 66.67 \\
Z(j\omega_r) = R &= 10\ \Omega
\end{aligned}$$

(b)

Using the expression for equivalent inductance of two coupled inductors L_1 and L_2, we get

$$\begin{aligned}
L_{eq} &= \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} \\
&= 0.5\ H
\end{aligned}$$

The circuit then becomes a C in parallel with $L_{eq} + R$.

$$\begin{aligned}
Z_{in} &= \left(\frac{s}{100} + \frac{1}{5 + 0.5s} \right)^{-1} \\
&= \frac{100}{\frac{100}{5+0.5s} + s}
\end{aligned}$$

For resonance, the denominator of Z_{in} must be real. This gives the following

$$\begin{aligned}
Im\left(\frac{100}{5 + 0.5j\omega_r} + j\omega_r \right) &= 0 \\
\Rightarrow \omega_r &= 10
\end{aligned}$$

$$\Rightarrow Z_{in}(j\omega_r) = 10\ \Omega$$

(c)

Using the expression for equivalent inductance of two coupled inductors L_1 and L_2, we get

$$\begin{aligned} L_{eq} &= \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} \\ &= 0.0588\ mH \end{aligned}$$

The circuit then becomes C in series with $(R||L_{eq})$.

$$Z_{in} = \frac{10^5}{s} + \frac{10}{1 + \frac{17\times 10^4}{s}}$$

For resonance, Z_{in} must be real. This gives the following

$$\begin{aligned} Im\left(\frac{10^4}{j\omega_r} + \frac{j\omega_r}{17\times 10^4 + j\omega_r}\right) &= 0 \\ \Rightarrow \omega_r &= 42500 \\ \Rightarrow Z_{in}(j\omega_r) &= 0.0588\ \Omega \end{aligned}$$

(d)

Let V be the voltage between the left terminals and I be the input current. Let the current I divide into I_1 flowing through L_2 and the rest flowing through R. Then following KVL equations can be written following the dot convention

$$\begin{aligned} V - L_1 I s - M(-I_1)s - L_2 I_1 s - M(-I)s &= 0 \\ -L_2 I_1 s - M(-I)s + (I - I_1)R &= 0 \\ \begin{bmatrix} (L_1 - M)s & (L_2 - M)s \\ R + Ms & -(R + L_2 s) \end{bmatrix} \begin{bmatrix} I(s) \\ I_1(s) \end{bmatrix} &= \begin{bmatrix} V(s) \\ 0 \end{bmatrix} \\ \Rightarrow Z &= \frac{V}{I} \end{aligned}$$

$$
\begin{aligned}
&= s\left[\frac{(L_2 - M)(R + Ms)}{R + L_2 s} + (L_1 - M)\right] \\
&= s
\end{aligned}
$$

Solution 62

(a)

The following KVL equations can be written following the dot convention

$$
\begin{aligned}
V_{in} - L_1 Is - M(-I)s - IR - L_2 Is + M(I)s &= 0 \\
V_{out} &= IR \\
\Rightarrow H(s) &= \frac{Vout}{V_{in}} \\
&= \frac{R}{(L_1 + L_2 - 2M)s + R} \\
&= \frac{16}{16 + s} \\
\Rightarrow h(t) &= 16e^{-16t}u(t)
\end{aligned}
$$

(b)

Assuming that current I flows clockwise in the loop containing R and L_1, we can write the following KVL equations following the dot convention. Note that no current flows through the second loop since it is open.

$$
\begin{aligned}
V_{in} - (R + L_1 s)I &= 0 \\
V_2 &= MIs \\
\Rightarrow H(s) = \frac{V_2}{V_{in}} &= \frac{Ms}{R + L_1 s} \\
&= \frac{0.15s}{600 + 0.3s} \\
V_{in} &= \frac{30}{s} \\
\Rightarrow V_2 &= V_{in} H(s) \\
&= \frac{15}{s + 2000} \\
\Rightarrow v_2(t) &= 15e^{-2000t}u(t)
\end{aligned}
$$

(c)

The circuit is (1/Cs) in parallel with $Z_{in}(s)$ in configuration 3.

$$\begin{aligned}
L &= L_3 + L_1 - \frac{M^2}{L_2 + L_4} \\
&= 2.8\ H \\
\Rightarrow V_{out} &= I_{in}\left(\frac{1}{Cs}||Ls\right) \\
&= 10\left(\frac{s}{200} + \frac{1}{2.8s}\right)^{-1} \\
&= \frac{2000s}{s^2 + 71.4286} \\
\Rightarrow v_{out}(t) &= 2000\cos(8.4515t)u(t)
\end{aligned}$$

(d)

The following KVL equations can be written following the dot convention,

$$\begin{aligned}
V_{in} - I_1 R_s - L_1 I_1 s - M(-I_2)s &= 0 \\
-L_2 I_2 s + M(I_1)s - R_L I_2 &= 0 \\
\begin{bmatrix} R_s + L_1 s & -Ms \\ -Ms & R_L + L_2 s \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} &= \begin{bmatrix} V_{in}(s) \\ 0 \end{bmatrix} \\
V_{out} &= I_2 R_L \\
\Rightarrow H(s) = \frac{V_{out}}{V_{in}} &= \frac{100s}{3s^2 + 400s + 10000} \\
\Rightarrow h(t) &= 50[e^{-100t} - (1/3)e^{-\frac{100}{3}t}]u(t)
\end{aligned}$$

Solution 63

(a)

The following KVL equations can be written following the dot convention,

$$\begin{aligned}
V_{in} - I_{out}R_s - L_1 I_{out}s - M(-I_{out})s - L_2 I_{out}s + M(I_{out})s - \frac{I_{out}}{Cs} &= 0 \\
\Rightarrow V_{in} - I_{out}R_s - I_{out}(L_1 + L_2 - 2M)s - \frac{I_{out}}{Cs} &= 0 \\
\Rightarrow L_{eq} &= L_1 + L_2 - 2M
\end{aligned}$$

$$= \quad 0.1\ H$$

(b)

$$\begin{aligned} H(s) &= \frac{I_{out}}{V_{in}} \\ &= \frac{1}{20 + 0.1s + 1000/s} \\ &= \frac{10s}{(s+100)^2} \end{aligned}$$

(c)

$$\begin{aligned} V_{step} &= \frac{H(s)}{s} \\ \Rightarrow v_{step}(t) &= 10te^{-100t}u(t) \end{aligned}$$

(d)

Compare the expression for $H(s)$ with the standard expression,

$$H(s) = K\frac{s}{s^2 + 2\sigma_p s + \omega_p^2}$$

Clearly, resonant frequency ω_p=(100/2π)=15.92 Hz

(e)

$$\begin{aligned} B_\omega = 2\sigma_p &= 200/(2\pi) \\ &= 31.83\ Hz \\ Q_{circuit} &= \frac{\omega_p}{B_\omega} \\ &= \frac{100}{200} = 0.5 \\ \omega_{1,2} &= \pm\sigma_p + \sqrt{\sigma_p^2 + \omega_p^2} \\ &= 41.42, 241.42 \end{aligned}$$

$$
\begin{aligned}
K &= 10 \\
\Rightarrow H_m &= \frac{|K|}{2\sigma_p} \\
&= 0.05
\end{aligned}
$$

(f)

$$
\begin{aligned}
K_f &= \frac{2000}{15.92} \\
&= 125.63 \\
R_{s,new} = R_s &= 20\ \Omega \\
L_{1,new} = \frac{L_1}{K_f} &= 0.0024\ H \\
L_{2,new} = \frac{L_2}{K_f} &= 0.0016\ H \\
C_{new} = \frac{C}{K_f} &= 8\ \mu F
\end{aligned}
$$

Solution 64

(a)

Here L_{eq} corresponds to configuration 1 which we have derived earlier.

$$
\begin{aligned}
L_{eq} &= L_1 - \frac{M^2}{L_2 + L_3} \\
&= 0.1\ H \\
Y_{in} &= \frac{1}{25} + \frac{s}{5000} + \frac{10}{s} \\
&= \frac{s^2 + 200s + 50000}{5000s} \\
\Rightarrow Z_{in} &= \frac{1}{Y_{in}} \\
&= \frac{5000s}{s^2 + 200s + 50000}
\end{aligned}
$$

(b)

$$
z = 0
$$

$$\begin{aligned}
p_{1,2} &= -100 \pm 200.00i \\
H(s) &= \frac{V_{out}}{I_{in}} \\
&= Z_{in} \\
\Rightarrow h(t) &= L^{-1}[Z_{in}] \\
&= 5000e^{-100t}\left[\cos(200t) - \frac{1}{2}\sin(200t)\right]u(t)
\end{aligned}$$

(c)

For the zero input response, $i_{in}(t) = 0$ and there is a current source of $CV_C(0-) = 2\ mA$ due to initial condition on the capacitor. Thus zero input response is given by

$$\begin{aligned}
V_{zi} &= 0.002Z_{in} \\
&= \frac{10s}{s^2 + 200s + 50000} \\
\Rightarrow v_{zi}(t) &= 10e^{-100t}\left[\cos(200t) - \frac{1}{2}\sin(200t)\right]u(t)
\end{aligned}$$

(d)

Zero state response is given by

$$\begin{aligned}
V_{zs} &= I_{in}Z_{in} \\
&= \left(\frac{s}{s^2 + 10000}\right)\left(\frac{5000s}{s^2 + 200s + 50000}\right) \\
\Rightarrow v_{zs}(t) &= \left[5\cos(100t) - 10\sin(100t) - 5e^{-100t}\left\{\cos(200t) - \frac{11}{2}\sin(200t)\right\}\right]u(t)
\end{aligned}$$

(e)

Complete response = zero input response + zero state response