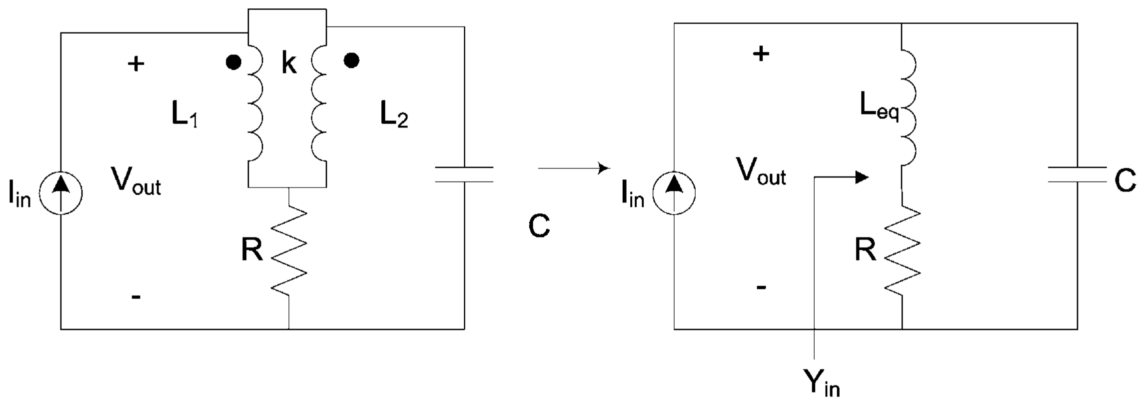
ECE 202 Linear Circuit Analysis II (Sp'09)

Homework #17 Solution (65–68)

Problem 65



$$\omega_r = 100\text{rad/sec}\ ,\ C = 0.7m\text{F}\ ,\ R = 2\Omega\ ,\ L_1 = L_2 = 0.2\text{H}$$

(a) L_{eq}, M, and k

To find the expression for the resonant frequency ω_r, it would be easier to find $Y_{in}(s)$ instead of $Z_{in}(s)$ in this case.

$$Y_{in}(s) = \frac{1}{L_{eq}s + R} + Cs$$

$$Y_{in}(j\omega) = j\omega C + \frac{1}{R + j\omega L_{eq}}\left(\frac{R - j\omega L_{eq}}{R - j\omega L_{eq}}\right)$$

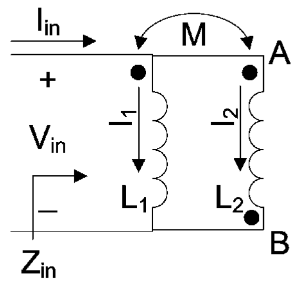
$$= \underbrace{\left(\frac{R}{R^2 + \omega^2 L_{eq}^2}\right)}_{Re[Y_{in}(j\omega)]} + j\,\underbrace{\omega\left(C - \frac{L_{eq}}{R^2 + \omega^2 L_{eq}^2}\right)}_{Im[Y_{in}(j\omega)]}$$

Setting the imaginary part equal to zero and solving for ω_r,

$$0 = C - \frac{L_{eq}}{R^2 + \omega_r^2 L_{eq}^2}$$

Plugging in the values and solving for L_{eq},we get $L_{eq} = 0.1400$ and 0.0029 [H].

Derivation of the equivalent inductance of two coupled inductors:



Writing two equations for $V_1(s)$ and $V_2(s)$ in a matrix form of $(Ax = b)$,

$$\begin{bmatrix} L_1 s & \pm M s \\ \pm M s & L_2 s \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where the positive and negative signs correspond to the dot in position A and B respectivly. Substituting $V_1(s) = V_2(s) = V_{in}$ and multiplying through by A^{-1}, we obtain $x = A^{-1}b$,

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{s^2(L_1L_2 - M^2)} \begin{bmatrix} L_2 s & \mp M s \\ \mp M s & L_1 s \end{bmatrix} \begin{bmatrix} V_{in} \\ V_{in} \end{bmatrix}$$

$$I_1 + I_2 = \frac{s(L_1 + L_2 \mp M \mp M)}{s^2(L_1L_2 - M^2)}$$

$$Z_{in}(s) = \frac{V_{in}}{I_1 + I_2} = s \underbrace{\left(\frac{L_1L_2 - M^2}{L_1 + L_2 \mp 2M} \right)}_{L_{eq}}$$

where -sign for dot in A, +sign for dot in B

For each value of L_{eq} we can solve for M value from the equation we obtained above.

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

And the coupling coefficient k can be calculated from

$$k \equiv \frac{M}{\sqrt{L_1L_2}}.$$

	$L_{eq} = 0.1400$	$L_{eq} = 0.0029$
M	0.200, 0.0800	0.200, −0.1942
k	1, 0.4	1, −

Eliminating $k = 1$ (ideal transformer) and negative mutual inductance cases, we are left with one answer choice.

$\boxed{L_{eq} = 0.1400[\text{H}]}$ $\boxed{M = 0.0800[\text{H}]}$ $\boxed{k = 0.4}$

(b) $H(s) = \dfrac{V_{out}(s)}{I_{in}(s)}$ and poles and zeros

Using MATLAB,

```
syms s R Leq C Yin Zin H Vout
Yin=s*C+1/(Leq*s+R);
H=1/Yin
% >>1/(s*C+1/(Leq*s+R))
pretty(simplify(H))
%                              Leq s + R
%                       ----------------------
%                        2
%                       s  C Leq + s C R + 1
R=2; C=0.7e-3; Leq=0.1400;
H=1/(s*C+1/(Leq*s+R))
pretty(simplify(H))
%                              7 s + 100
%                 10000 ----------------------
%                           2
%                       49 s  + 700 s + 500000
num=10000*[7 100];
den=[49 700 500000];
z=roots(num)
% z =
%   -14.2857
p=roots(den)
% p =
% 1.0e+002 *
%   -0.0714 + 1.0076i
%   -0.0714 - 1.0076i
```

$$
\begin{aligned}
H(s) = Z_{in}(s) &= \frac{1}{Y_{in}} \\
&= \frac{1}{\dfrac{1}{L_{eq}s + R} + Cs} \\
&= \frac{\dfrac{1}{C}\left(s + \dfrac{R}{L_{eq}}\right)}{s^2 + \left(\dfrac{R}{L_{eq}}\right) + \dfrac{1}{L_{eq}C}} \\
&= \boxed{1428.57\frac{s + 14.2857}{s^2 + 14.2857s + 10204}}
\end{aligned}
$$

$$\text{Zeros: } \boxed{z_1 = -14.2857}$$
$$\text{Poles: } \boxed{p_{1,2} = -7.14 \pm j100.76}$$

(c) Circuit Q, Q_{cir}

$$Q_p = \frac{\omega_p}{2\sigma_p}$$

$$Q_{cir} \approx Q_p = \frac{\omega_p}{B_w} = \frac{\sqrt{10204}}{14.2857} = \boxed{7.071}$$

(d) B_w [Hz]

The transfer function we have has a pair of complex poles and a zero off the origin. As stated in the textbook, the exact expression for the bandwidth is not available. To find the exact bandwidth, we use MATLAB.

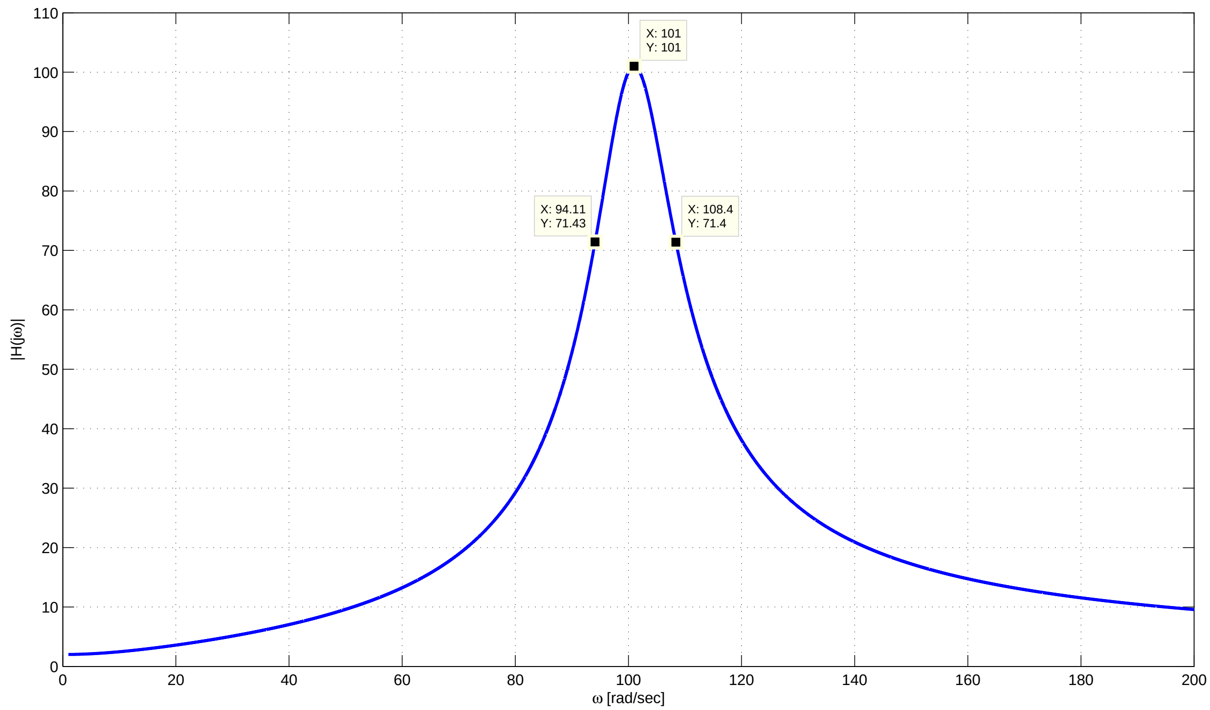
```
clear all;
num=10000*[7 100];
den=[49 700 500000];
w=linspace(1,10e2, 10e4);
h=freqs(num,den,w);
plot(w,abs(h),'linewidth',2)
axis([0 200 0 110]); xlabel('\omega [rad/sec]'); ylabel('|H(j\omega)|');
grid on;
Hmax=abs(max(h))
H3db=Hmax/sqrt(2)
%Hmax =
%       100.9951
%H3db =
%       71.4144
```

The above code will generate a plot for part (g) and give the values for H_m and the H_{3dB}. Using those values, we can find the half power frequencies and eventually the exact bandwidth from the generated plot.
From the plot, we have

$$\omega_1 = 94.11[\text{rad/s}]$$
$$\omega_2 = 108.4[\text{rad/s}]$$

Therefore the exact bandwidth is:

$$B_{w,exact} = 108.4 - 94.11 = 14.29[\text{rad/s}] \Rightarrow \boxed{2.2743[\text{Hz}]}$$



The approximate Bandwidth is:

$$B_{w,approx} = 2\sigma_p = 14.2857[\text{rad/sec}] \Rightarrow \boxed{2.2736[\text{Hz}]}$$

(e) approximate H_m and ω_m

$$|H(j\omega)|_{max} \approx \frac{|K|}{2\sigma_p} = \frac{1428.6}{14.286} = \boxed{100[\Omega]}$$

$$\omega_m \approx \omega_p = \sqrt{10204} = 101.0149[\text{rad/s}] \Rightarrow \boxed{16.0770[\text{Hz}]}$$

(f) $\omega_{1,2}$ [Hz] and the percentage errors

$$\omega_{1,approx} = \omega_m - \frac{B_w}{2} = 93.8720[\text{r/s}] \Rightarrow \boxed{14.94[\text{Hz}]}$$
$$\omega_{2,approx} = \omega_m + \frac{B_w}{2} = 108.1577[\text{r/s}] \Rightarrow \boxed{17.2138[\text{Hz}]}$$

$$\omega_1 = 94.11[\text{r/s}] \Rightarrow \boxed{14.9781[\text{Hz}]}$$
$$\omega_2 = 108.4[\text{r/s}] \Rightarrow \boxed{17.2524[\text{Hz}]}$$

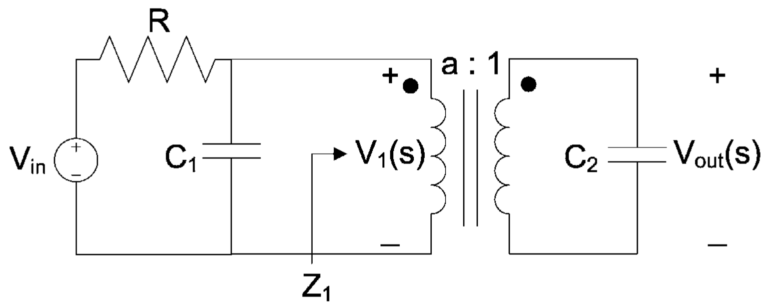
$$\%\text{ error} = \frac{|\omega_{1/2} - \omega_{1/2,approx}|}{\omega_{1/2}} \times 100$$
$$\%\text{ error for } \omega_1 = \boxed{0.2529\%}$$
$$\%\text{ error for } \omega_2 = \boxed{0.2235\%}$$

(g) plot of $|H(j\omega)|$ vs. ω

Done in part (d)

Problem 66

(a)



$$C_1 = 0.1\text{F},\ C_2 = 0.001\text{F},\ R = 2\Omega,\ v_{in}(t) = \delta(t) \rightarrow V_{in}(s) = 1$$
$$v_{out}(t) = 25e^{-2.5t}u(t) \Rightarrow V_{out}(s) = \frac{25}{s+2.5}$$

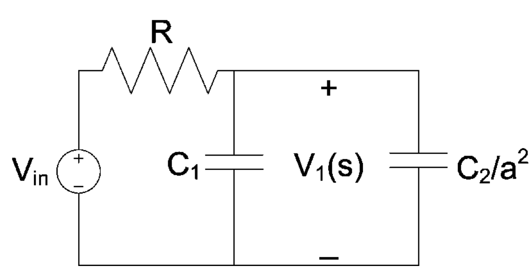
Z_1 seen from the left side of the ideal transformer is

$$Z_1 = a^2\frac{1}{C_2 s} = \frac{1}{s\frac{C_2}{a^2}}$$

and the voltages are related by

$$\frac{V_{out}(s)}{V_1} = \frac{1}{a} \Rightarrow V_1 = aV_{out}$$

We can redraw the circuit as below:



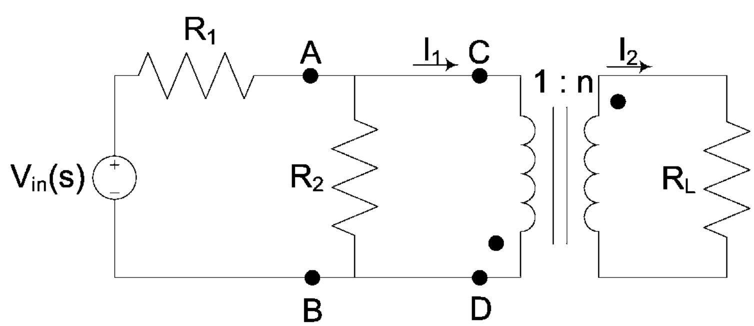
Solving for V_1 using simple voltage division,

$$V_1 = \frac{1/\left(C_1 + C_2/a^2\right)s}{R + 1/\left(C_1 + C_2/a^2\right)s} V_{in}$$
$$= \frac{\frac{1}{R(C_1 + C_2/a^2)}}{s + \frac{1}{R(C_1 + C_2/a^2)}}$$
$$V_{out} = \frac{1}{a}\frac{\frac{1}{R(C_1 + C_2/a^2)}}{s + \frac{1}{R(C_1 + C_2/a^2)}} = \frac{1}{0.1}\frac{2.5}{s + 2.5}$$

Matching the coefficients,

$$2.5 = \frac{1}{R(C_1 + C_2/a^2)}$$
$$a = \boxed{0.1}$$

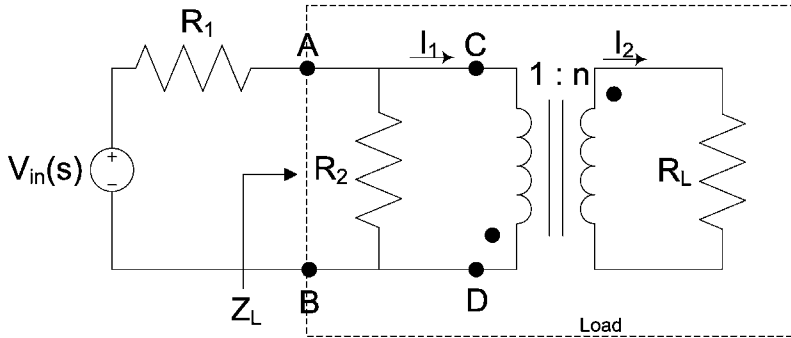
(b)



$R_1 = 25\Omega$, $R_2 = 50\Omega$, $R_L = 2\Omega$

For maximum power transfer, the load impedance, Z_L should match the source impedance.

(i) When the load is to the right of A-B



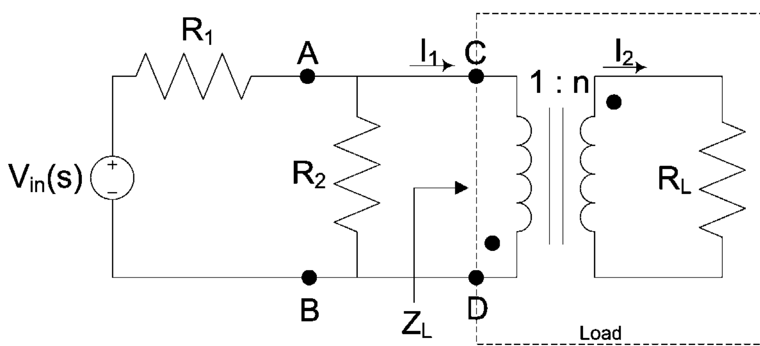
For maximum power transfer,

$$\begin{aligned}\text{source impedance} &= \text{load impedance}\\ R_1 &= Z_L\\ &= R_2 \| \left(\frac{1}{n}\right)^2 R_L\\ &= \frac{R_2 R_L}{n^2 R_2 + R_L}\end{aligned}$$

Solving for n,

$$n = \sqrt{\frac{R_L(R_2 - R_1)}{R_1 R_2}} = \sqrt{\frac{2(50-25)}{(25)(50)}} = \boxed{\frac{1}{5}}$$

(ii) When the load is to the right of C-D

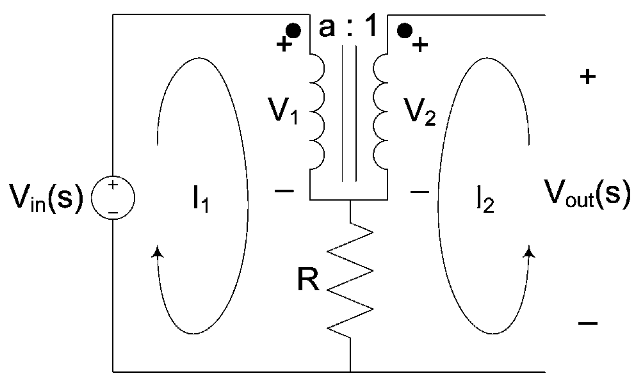


$$\begin{aligned}\text{source impedance} &= \text{load impedance}\\ R_1 \| R_2 &= Z_L\\ \frac{R_1 R_2}{R_1+R_2} &= \left(\frac{1}{n}\right)^2 R_L\\ &= \frac{R_L}{n^2}\end{aligned}$$

Solving for n,

$$n = \sqrt{\frac{R_L(R_1+R_2)}{R_1 R_2}} = \sqrt{\frac{2(25+50)}{(25)(50)}} = \boxed{0.3464}$$

(c) Thevenin equivalent in terms of a and R



Writing two loop equations and two equations for ideal transformer, we get

$$\begin{aligned}-V_{in} + V_1 + R(I_1+I_2) &= 0\\ -V_{out} + V_2 + R(I_2+I_1) &= 0\\ V_1 - aV_2 &= 0\\ I_1 + I_2/a &= 0\end{aligned}$$

Writing in a matrix form,

$$\begin{bmatrix}1 & 0 & R & R\\ 0 & 1 & R & R\\ 1 & -a & 0 & 0\\ 0 & 0 & 1 & 1/a\end{bmatrix}\begin{bmatrix}V_1\\ V_2\\ I_1\\ I_2\end{bmatrix} = \begin{bmatrix}V_{in}\\ V_{out}\\ 0\\ 0\end{bmatrix}$$

To find the Thevenin equivalent circuit ($V_{oc} = Z_{th} I_{sc}$), we need to find any two of the three terms. Using the above matrix equations in the form of $Ax = b$, we can find Z_{th} and I_{sc} in the following manner.

Thevenin Impedance, Z_{th} (Impedance of a dead circuit)

$$Z_{th} = \frac{V_{out}}{I_2} \text{ with } V_{in} = 0$$

Solving the below matrix equation for I_2,

$$\begin{bmatrix} 1 & 0 & R & R \\ 0 & 1 & R & R \\ 1 & -a & 0 & 0 \\ 0 & 0 & 1 & 1/a \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ V_{out} \\ 0 \\ 0 \end{bmatrix}$$

$$I_2 = \frac{a^2}{R(a-1)^2} V_{out}$$

$$Z_{th} = \frac{V_{out}}{I_2} = \frac{R(a-1)^2}{a^2}$$

Short Circuit current, I_{sc}

$$I_{sc} = -I_2 \text{ with } V_{out} = 0$$

Solving the below matrix equation for I_2,

$$\begin{bmatrix} 1 & 0 & R & R \\ 0 & 1 & R & R \\ 1 & -a & 0 & 0 \\ 0 & 0 & 1 & 1/a \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_{in} \\ \mathbf{0} \\ 0 \\ 0 \end{bmatrix}$$

$$-I_2 = I_{sc} = \frac{a}{R(a-1)^2} V_{in}$$

The MATLAB code used:

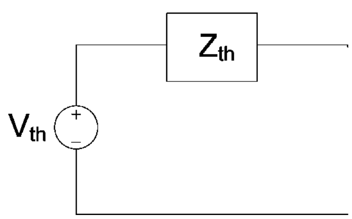
```
syms V1 V2 Vin Vout I1 I2 R a Zth Isc A x1 x2 b1 b2

A=[ 1 0 R R; 0 1 R R; 1 -a 0 0 ; 0 0 1 1/a];
b1=[0; Vout; 0; 0];
b2=[Vin; 0; 0; 0];

x1=inv(A)*b1

% x1 =
            %     -1/(-1+a)*a*Vout
            %       -1/(-1+a)*Vout
```

```
            % -1/R/(-1+a)^2*a*Vout
            %  a^2/R/(-1+a)^2*Vout
pretty(simplify(a^2/R/(-1+a)^2*Vout))
%                         2
%                        a  Vout
%                      -----------
%                                2
%                      R (-1 + a)
x2=inv(A)*b2
% x2 =
            %      1/(-1+a)*a*Vin
            %        1/(-1+a)*Vin
            %    1/R/(-1+a)^2*Vin
            % -1/R/(-1+a)^2*a*Vin
pretty(simplify(-1/R/(-1+a)^2*a*Vin))
%
            %                         a Vin
            %                    - -----------
            %                                2
            %                      R (-1 + a)
```

Using the relationship $V_{th} = Z_{th} I_{sc}$ we find V_{th} to be,

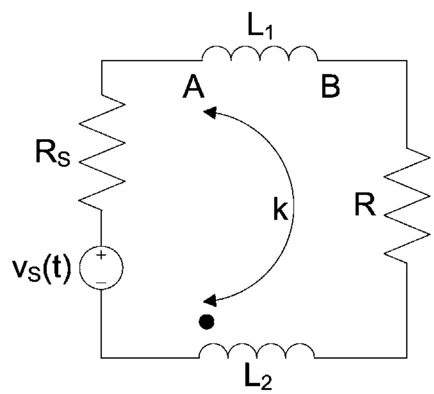
$$V_{th} = Z_{th} I_{sc} \quad = \frac{R(a-1)^2}{a^2} \frac{a}{R(a-1)^2} V_{in} = \frac{1}{a} V_{in}$$

The Thevenin Equivalent Circuit is:



$$\boxed{V_{th} = \frac{1}{a} V_{in}} \text{ and } \boxed{Z_{th} = \frac{R(a-1)^2}{a^2}}$$

Problem 67 Maximum instantaneous steady state power



$$L_1 = 0.8\text{H},\ L_2 = 0.45\text{H},\ M = 0.175\text{H},\ R_S = R = 12\Omega$$
$$v_s(t) = V_m \cos\omega t \Rightarrow 30\cos 10t$$

Maximum instantaneous steady state power of the resistor would be $P_{R,max} = \dfrac{v_{R,max}^2}{R}$
Writing a loop equation,

$$V_s(s) = I\left(L_1 s \mp Ms + R + L_2 s \mp Ms\right)$$
$$\Rightarrow V_R = IR = \frac{R}{(R_S + R) + (L_1 + L_2 \mp 2M)s} V_s$$
$$\Rightarrow H(s) = \frac{V_R}{V_s} = \frac{R}{(R_S + R) + (L_1 + L_2 \mp 2M)s}$$

where minus and plus signs in front of Ms corresponds to dot position in A and B respectively.

Dot A

$$H_a(s) = \frac{12}{12 + 12 + (0.8 + 0.45 - 2(0.175))}$$
$$V_R(s) = H_a(s)V_s = \frac{12}{24 + s(0.9)} V_s$$
$$= \frac{40/3}{s + 80/3} Vs$$

For $v_s(t) = 30\cos 10t$,

$$\begin{aligned} v_R(t) &= 30|H_a(j10)|\cos(10t + \angle(H(j10))) \\ &= 30\frac{4}{\sqrt(73)}\cos(10t+\theta) = (30)(0.4682)\cos(10t+\theta) \end{aligned}$$

$$v_{R,max} = 14.0449 \text{ [V]}$$

$$\begin{aligned} P_{R,max} &= \frac{(14.0449)^2}{12} \\ &= \boxed{16.4383[\text{W}]} \end{aligned}$$

Dot B

$$\begin{aligned} H_b(s) &= \frac{12}{12 + 12 + (0.8 + 0.45 + 2(0.175))} \\ V_R(s) &= H_b(s)V_s = \frac{12}{24 + s(1.6)}V_s \\ &= \frac{15/2}{s+15}V_s \end{aligned}$$

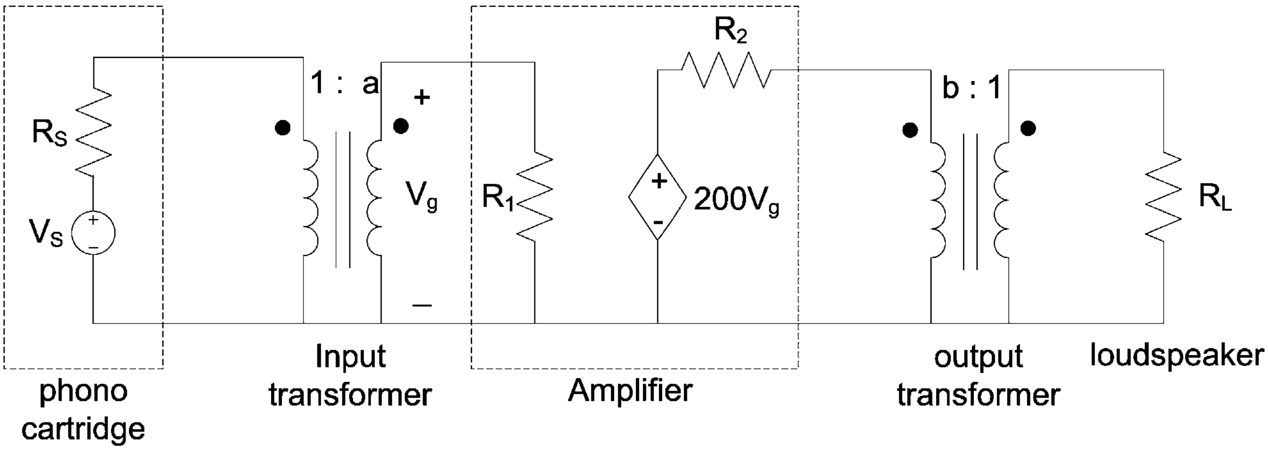
For $v_s(t) = 30\cos 10t$,

$$\begin{aligned} v_R(t) &= 30|H_b(j10)|\cos(10t + \angle(H(j10))) \\ &= 30(0.4160)\cos 10t + \theta \end{aligned}$$

$$v_{R,max} = 12.480 \text{ [V]}$$

$$\begin{aligned} P_{R,max} &= \frac{(12.480)^2}{12} \\ &= \boxed{12.9792[\text{W}]} \end{aligned}$$

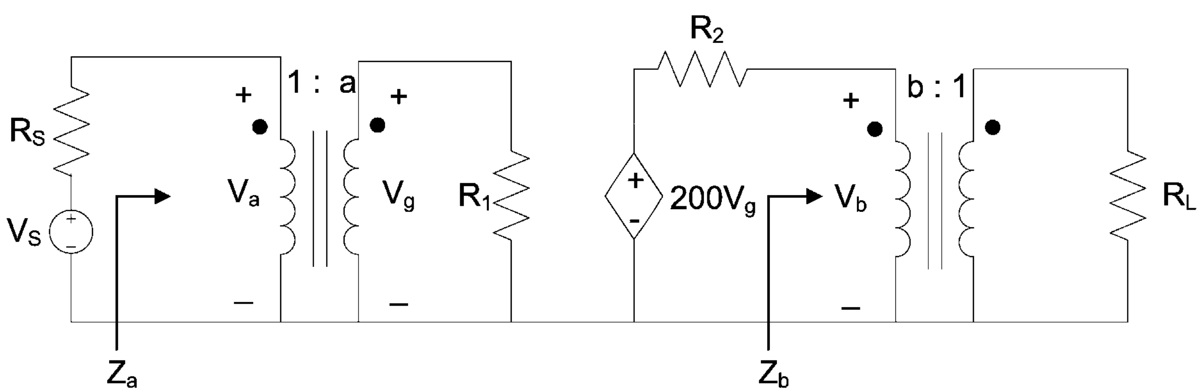
Problem 68



$$R_S = 25\Omega,\ R_1 = 1.6k\Omega,\ R_2 = 200\Omega,\ R_L = 4\Omega$$
$$v_s(t) = 0.1\sqrt{2}\cos(\omega t)\ [\text{V}] \Rightarrow v_{s,rms} = 0.1\ [\text{V}]$$

(a)

Following the condition for the maximum power transfer, we can match the impedances at input and output transformers.



At the input transformer, the maximum power transfer occurs when

$$R_s = Z_a = \left(\frac{1}{a}\right)^2 R_1$$
$$\Rightarrow a^2 = \frac{R_1}{R_s}$$
$$\Rightarrow a = \sqrt{\frac{R_1}{R_s}} = \sqrt{\frac{16k}{25}} = 8$$

At the output transformer, the maximum power transfer occurs when

$$R_2 = Z_b = \left(\frac{b}{1}\right)^2 R_L$$
$$\Rightarrow b^2 = \frac{R_2}{R_L}$$
$$\Rightarrow b = \sqrt{\frac{R_2}{R_L}} = \sqrt{\frac{200}{4}} = \sqrt{50} = 7.07$$

$$\therefore \boxed{a = 8} \qquad \text{and} \qquad \boxed{b = \sqrt{50} = 7.07}$$

The voltages of the transformer are:

$$V_a = \frac{V_{s,rms}}{2}$$
$$V_g = aV_a = a\frac{V_{s,rms}}{2}$$
$$V_b = \frac{200V_g}{2} = 100(aV_a) = 100a\frac{V_{s,rms}}{2} = 50aV_{s,rms}$$
$$V_{out} = 50\left(\frac{8}{\sqrt{50}}\right)V_{s,rms}$$
$$= \frac{40}{\sqrt{50}}$$

$$P_{max,4\Omega} = \frac{V_{out}^2}{R_L} = \left(\frac{40}{\sqrt{50}}\right)^2 \frac{1}{4} = \boxed{8[\text{W}]}$$

(b)

$R_L \Rightarrow 16\Omega$
This will change the voltage values of the right half of the circuit (output transformer).

$$
\begin{aligned}
Z_b &= b^2 R_L = (\sqrt{50})^2(16) = 800\Omega \\
V_b &= \frac{800}{800+200} 200 V_g \qquad \text{where} \qquad V_g = \frac{a}{2} V_{s,rms} \\
&= \frac{4}{5} 200 \left(\frac{a}{2} V_{s,rms}\right) = 80 a V_{s,rms} \\
V_{out} &= \frac{1}{b} V_b = 80 \frac{a}{b} V_{s,rms} \\
&= 80 \frac{8}{\sqrt{50}} 0.1 = \frac{64}{\sqrt{50}}
\end{aligned}
$$

$$
P_{max,16\Omega} = \frac{V_{out}^2}{R_L} = \left(\frac{64}{\sqrt{50}}\right)^2 \frac{1}{16} = \boxed{5.12[\mathrm{W}]}
$$