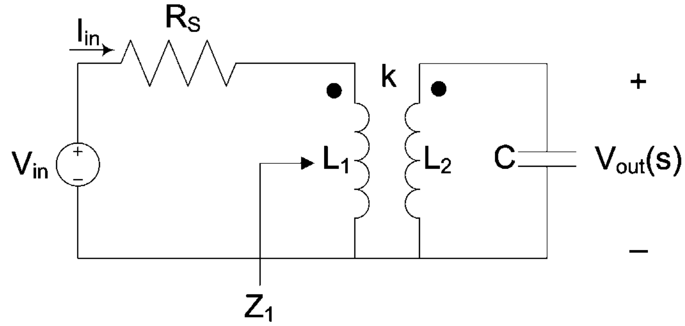
ECE 202 Linear Circuit Analysis II (Sp'09)

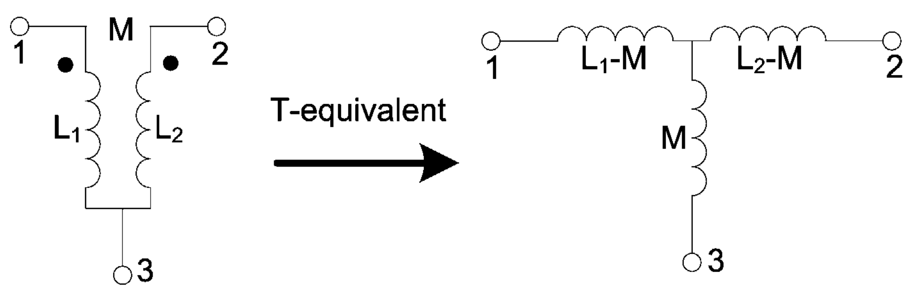
Homework #18 Solution (69–72)

Problem 69

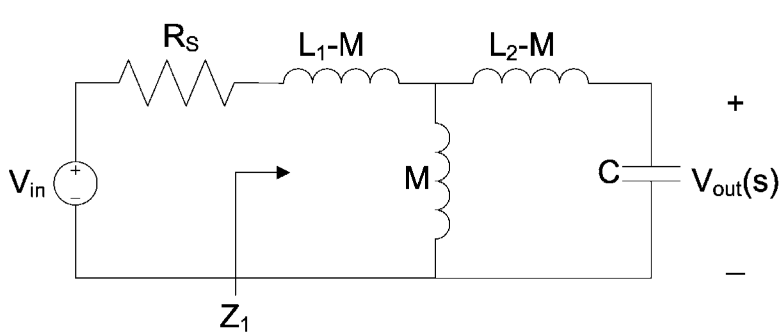


$$R_s = 0.1\Omega\ ,\ L_1 = 2\text{H}\ ,\ L_2 = 1\text{H}\ ,C = 0.02\text{F}\ ,\ \omega_r = 10\text{rad/sec}$$

T-equivalent circuit



Using T-equivalent circuit, the circuit can be redrawn as



(a) Find k

Given the resonant frequency value we will get an expression for the resonant frequency and find the value for k. To find the expression for the resonant frequency we need to compute either $Z_{in}(s)$ or $Y_{in}(s)$. We can use both the original circuit and T-equivalent circuit to compute Z_{in}. Here, we will show both ways.

From the orginial circuit drawing,
Define I_2 as the current flowing into the dotted terminal of L_2 and we can write the loop equations as following:

$$\text{Loop1}: V_{in} = (R_s + L_1 s)I_{in} + (Ms)I_2 \quad (69.1)$$

$$\text{Loop2}: 0 = (Ms)I_{in} + \left(L_2 s + \frac{1}{Cs}\right) I_2 \quad (69.2)$$

Since $Z_{in} = \dfrac{V_{in}}{I_{in}}$ we can simply divide both (eq.69.1) and (eq.69.2) by I_{in}. Then

$$\text{Loop1}: \frac{V_{in}}{I_{in}} = (R_s + L_1 s) + (Ms)\frac{I_2}{I_{in}} \quad (69.3)$$

$$\text{Loop2}: 0 = (Ms) + \left(L_2 s + \frac{1}{Cs}\right) \frac{I_2}{I_{in}} \quad (69.4)$$

From (eq.69.4) we get $\dfrac{I_2}{I_{in}} = -\dfrac{Ms}{\frac{1}{Cs} + L_2 s}$. Substituting into (eq.69.3) gives

$$Z_{in}(s) = \frac{V_{in}}{I_{in}} = R_s + L_1 s - \frac{M^2 C s^3}{1 + L_2 C s^2}$$

$$Z_{in}(j\omega) = R_s + j\omega L_1 + j\omega \frac{M^2 C \omega^2}{1 - L_2 C \omega^2}$$

$$Im[Z_{in}(j\omega)] = j\omega \left(L_1 + \frac{M^2 C \omega^2}{1 - L_2 C \omega^2}\right)$$

At resonant frequency, Im$[Z_{in}(j\omega_r)]$=0.

$$0 = L_1 + \frac{M^2 C \omega_r^2}{1 - L_2 C \omega_r^2}$$

$$0 = 2 + \frac{M^2 (0.02)(10^2)}{1 - (1)(0.02)(10^2)}$$

$$\Rightarrow M = 1$$

From the circuit drawing with T-equivalent inductors,

$$\begin{aligned} Z_1(s) =& (L_1 - M)s + \left(Ms \| \left((L_2 - M)s + \frac{1}{Cs} \right) \right) \\ =& (L_1 - M)s + \frac{(Ms)\left((L_2 - M)s + \frac{1}{Cs} \right)}{Ms + (L_2 - M)s + \frac{1}{Cs}} \\ =& (L_1 - M)s + \frac{Ms((L_2 - M)Cs^2 + 1)}{L_2Cs^2 + 1} \\ Z_1(j\omega) =& j\omega(L_1 - M) + \frac{j\omega M(1 - (L_2 - M)C\omega^2)}{1 - L_2C\omega^2} \end{aligned}$$

At resonant frequency, $Im[Z(j\omega_r)] = 0$

$$j\omega \left[(L_1 - M) + \frac{M(1 - (L_2 - M)C\omega_r^2)}{1 - L_2C\omega_r^2} \right] = 0$$

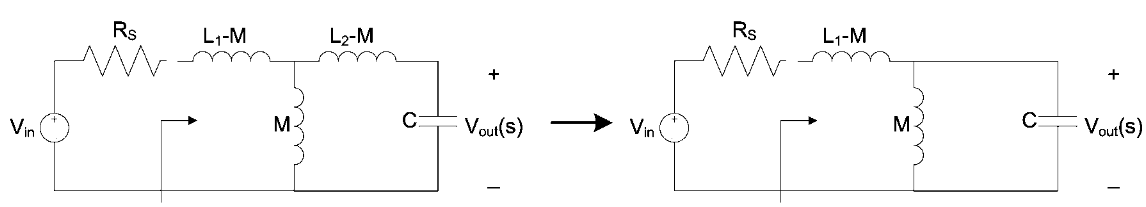
Plugging in $L_1 = 2$H, $L_2 = 1$H, $C = 0.02$F, and $\omega_r = 10$r/s,

$$\begin{gathered} (2 - M) + \frac{M(1 - 2(1 - M))}{1 - 2} = 0 \\ (2 - M) - M(2M - 1) = 0 \\ \boxed{M = 1} \\ \Rightarrow k = \frac{M}{\sqrt{L_1L_2}} = \frac{1}{\sqrt{2}} = \boxed{0.707} \end{gathered}$$

(b) $H(s) = \dfrac{V_{out}(s)}{I_{in}(s)}$ and poles and zeros

From section (a) we found that $M = 1$H and $L_2 - M = 0$ which corresponds to zero impedance → short circuit.

From the right figure we find



$$
\begin{aligned}
V_{out} =& \frac{Ms||1/Cs}{R_s + (L_1 - M)s + (Ms||1/Cs)} V_{in} \\
=& \frac{\frac{Ms}{MCs^2+1}}{R_s + (L_1 - M)s + \frac{Ms}{MCs^2+1}} V_{in} \\
=& \frac{Ms}{((L_1 - M)MC)s^3 + (R_s MC)s^2 + L_1 s + R_s} V_{in} \\
=& \frac{(\frac{1}{(L_1 - M)C})s}{s^3 + (\frac{R_s}{L_1 - M})s^2 + (\frac{L_1}{(L_1 - M)MC})s + (\frac{R_s}{(L_1 - M)MC})}
\end{aligned}
$$

Plugging in the component values,

$$
\begin{aligned}
V_{out} =& \frac{50s}{s^3 + 0.1s^2 + 100s + 5} V_{in} \\
H(s) =& \frac{V_{out}}{V_{in}} = \boxed{\frac{50s}{s^3 + 0.1s^2 + 100s + 5}}
\end{aligned}
$$

$$
\boxed{z = 0}
$$

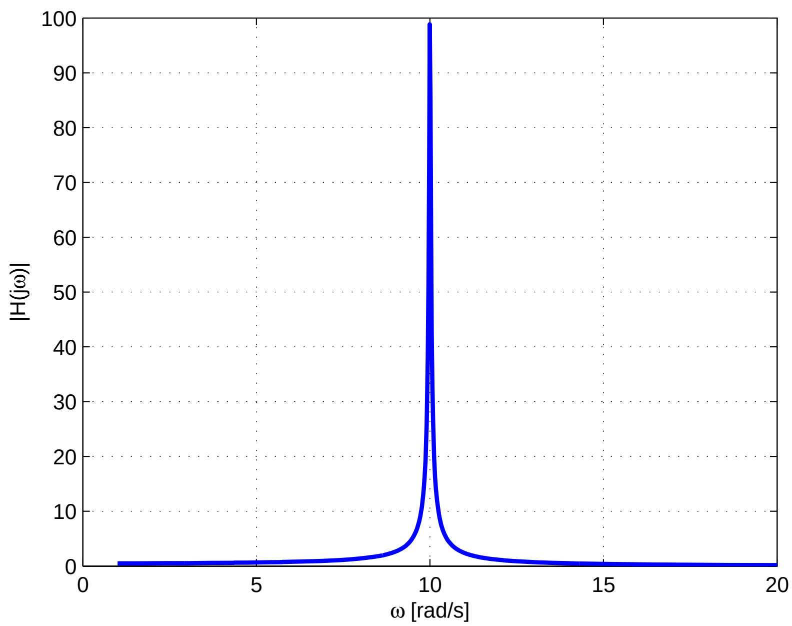
$$
\boxed{p_{1,2} = -0.0250 \pm j9.9998}, \boxed{p_3 = -0.05}
$$

```
syms s M L1 L2 Rs H Z1 C
%
Z1=(M*s*1/(C*s))/(M*s+1/(C*s))
H=Z1/(Rs+(L1-M)*s+Z1)
% H =
%
% M/C/(M*s+1/C/s)/(Rs+(L1-M)*s+M/C/(M*s+1/C/s))
M=1; L1=2; C=0.02; Rs=0.1;
H=M/C/(M*s+1/C/s)/(Rs+(L1-M)*s+M/C/(M*s+1/C/s))
% H =
%
% 50/(s+50/s)/(1/10+s+50/(s+50/s)
pretty(simplify(H))
%                                       s
%                          500 ------------------------
%                                 2             3
%                                s  + 50 + 10 s  + 1000 s
num=[500 0];
den=[10 1 1000 50];
```

```
poles=roots(den)
% poles =
%
%   -0.0250 + 9.9998i
%   -0.0250 - 9.9998i
%   -0.0500
firstorder=poly(poles(3))
% firstorder =
%
%     1.0000    0.0500
secondorder=poly(poles(1:2))
% secondorder =
%
%     1.0000    0.0500   99.9975
wm=sqrt(secondorder(3))
% wm =
%
%     9.9999
w=linspace(1,20,1000);
h=freqs(num,den,w);
plot(w,abs(h),'linewidth',2); grid on;
xlabel('\omega [rad/s]'); ylabel('|H(j\omega)|');
```

(c)MATLAB plot

The MATLAB code in part (b) generates the plot from which we see a bandpass characteristic behavior.



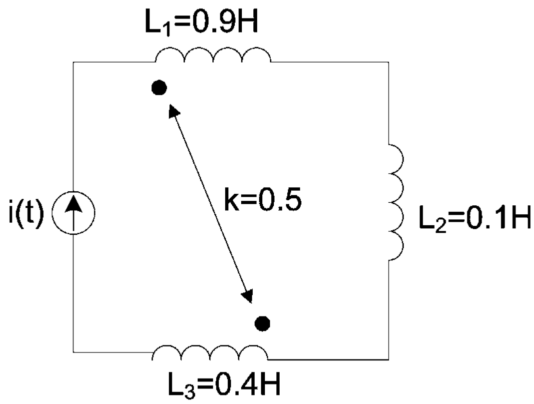
(d) ω_m, B_w, Q, H_m, ω_1 and ω_2

From the MATLAB code in part (b),

$$\begin{aligned} H(s) =& H_1(s)H_2(s) = \frac{50s}{s^3 + 0.1s^2 + 100s + 5} \\ =& \left(\frac{10}{s + 0.05}\right)\left(\frac{5s}{s^2 + 0.05s + 99.9975}\right) \end{aligned}$$

$$\begin{aligned} \omega_m &\approx \sqrt{99.9975} = 9.9999 \approx \boxed{10[\text{rad/s}]} \\ B_w &\approx \boxed{0.05[\text{rad/s}]} \\ Q &= \frac{\omega_m}{B_w} = \frac{10}{0.05} = \boxed{200} \\ H_m &\approx \frac{5}{0.05} = \boxed{100[\text{V/V}]} \\ \omega_1 &\approx \omega_m - \frac{B_w}{2} = \boxed{9.975[\text{rad/s}]} \\ \omega_2 &\approx \omega_m + \frac{B_w}{2} = \boxed{10.025[\text{rad/s}]} \end{aligned}$$

Problem 70 Maximum Instantaneous stored energy



$$i(t) = 2\sqrt(2)\cos(100t + 35°)\text{A}$$

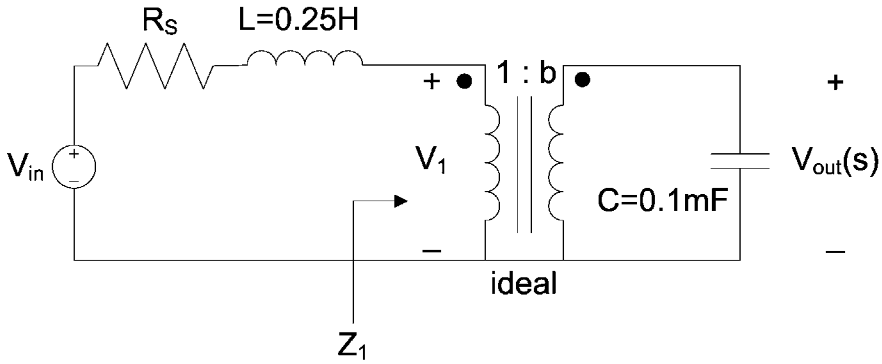
$$\begin{aligned} k = \frac{M}{\sqrt{L_1 L_2}} \Rightarrow M &= k\sqrt{L_1 L_2} \\ &= 0.3 \end{aligned}$$

$$\begin{aligned} Z_{eq} =& s(L_1 + L_2 + 2M + L3) \\ \Rightarrow L_{eq} =& L_1 + L_2 + 2M + L3 \\ =& 0.9 + 0.4 + 2(0.3) + 0.1 \\ =& 2\text{H} \end{aligned}$$

$$I_{max} = 2\sqrt(2)$$

$$\begin{aligned} W_{max} =& \frac{1}{2} L_{eq} I_{max}^2 \\ =& \frac{1}{2}(2)(2\sqrt(2))^2 \\ =& \boxed{8\text{J}} \end{aligned}$$

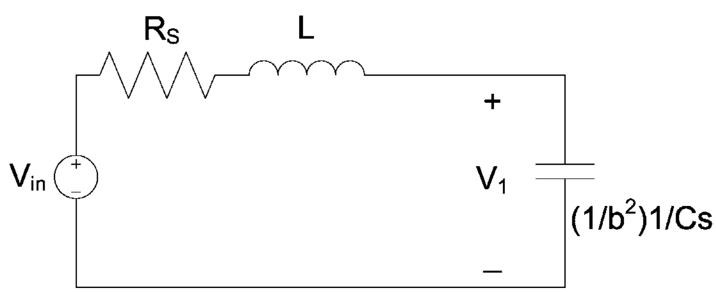
Problem 71



$$\text{where } Z_1 = \left(\frac{1}{b}\right)^2 \frac{1}{Cs} = \frac{1}{(b^2C)s}$$

(a) Transfer function

Using impedance transformation property,



$$V_1 = \frac{\frac{1}{b^2Cs}}{R_s + Ls + \frac{1}{b^2Cs}} V_{in}$$

$$V_1 = \frac{1}{b} V_{out}$$

Combining the above two equations,

$$\frac{V_{out}}{V_{in}} = H(s) = \frac{\frac{1}{bLC}}{s^2 + \frac{R_s}{L}s + \frac{1}{b^2LC}}$$
$$= \boxed{\frac{4000}{s^2 + 0.8s + 400}}$$

The pole locations are:

$$p_{1,2} = \sigma_p \pm j\omega_d$$
$$p_{1,2} = -0.4 \pm j19.9960$$

$-\sigma_p$ gives the damping factor and ω_d gives the oscillation frequency.
From the location of the poles (two distinct complex roots), we see that the impulse and step responses will be underdamped.

Using MATLAB,

```
syms s Vin Vout V1 V2 Rs L C Z1 b H yimpulse ystep
Z1=(1/b)^2*(1/(C*s));
V1=Z1/(Rs+L*s+Z1)*Vin;
H=Z1/(Rs+L*s+Z1)*b
% H =
%
% 1/b/C/s/(Rs+L*s+1/b^2/C/s)
pretty(simplify(1/b/C/s/(Rs+L*s+1/b^2/C/s)))
%                                        b
%                             -------------------------
%                                   2         2  2
%                             Rs s b  C + L s  b  C + 1

Rs=0.2; L=0.25; C=0.1e-3; b=10;
H=1/b/C/s/(Rs+L*s+1/b^2/C/s)
pretty(simplify(H))
%                                     20000
%                                 -----------------
%                                        2
%                                 4 s + 5 s  + 2000
num=[20000];
den=[5 4 2000];
poles=roots(den)
% poles =
%
%   -0.4000 +19.9960i
%   -0.4000 -19.9960i

%Part b)
%Impulse response
```

```
yimpulse=ilaplace(H);
pretty(simplify(ilaplace(H)))
%                    10000   1/2                          1/2
%                    ----- 51    exp(- 2/5 t) sin(14/5 51    t)
%                     357

%Step response
ystep=ilaplace(H/s);
pretty(simplify(ilaplace(H*(1/s))))
%                                    1/2
%   10 - 10 exp(- 2/5 t) cos(14/5 51    t)
%
%            10    1/2                         1/2
%          - --- 51    exp(- 2/5 t) sin(14/5 51    t)
%            357

wp=abs(poles(1))
% wp =
%
%    20.0000
sigmag=abs(real(poles(1)))
% sigmag =
%
%     0.4000
Q=wp/(2*sigmap)
% Q =
%
%    25.0000
w=linspace(1,40,1000);
h=freqs(num,den,w);
plot(w,abs(h),'linewidth',2); grid on;
xlabel('\omega [rad/s]'); ylabel('|H(j\omega)|');
```

(b) Impulse and Step response

The impulse and step responses are easily calculated using MATLAB command *ilaplace*. From the above MATLAB code we found that the impulse and step responses are:

$$\text{Impulse Response} = \boxed{200e^{-0.4t}\sin(20t)u(t)}$$

$$\text{Step Response} = \boxed{[10 - 9.9960e^{-0.4t}\cos(20t) - 0.1999e^{-0.4t}\sin(20t)]u(t)}$$

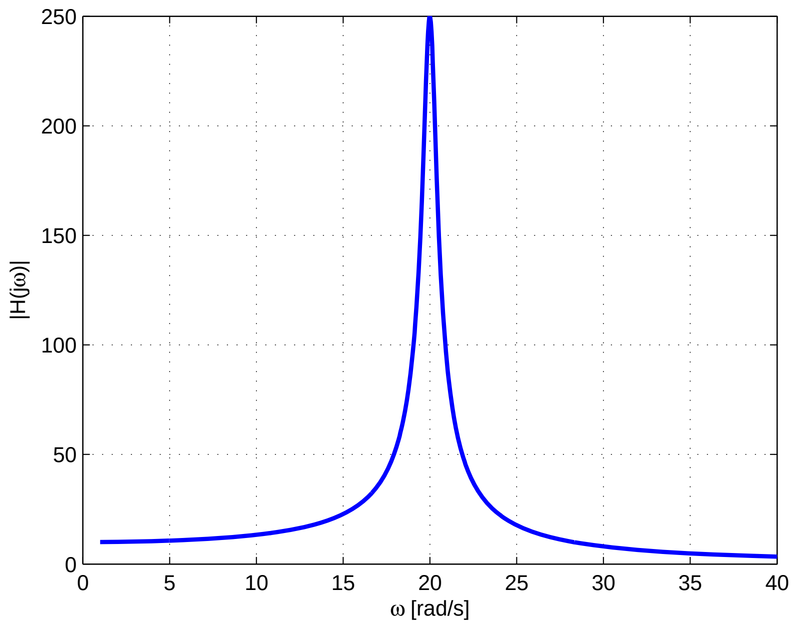
(c) ω_p, Q_p, ω_m, B_w, $\omega_{1,2}$

$$p_{1,2} = \sigma_p \pm j\omega_d = 0.4 \pm j19.9960$$

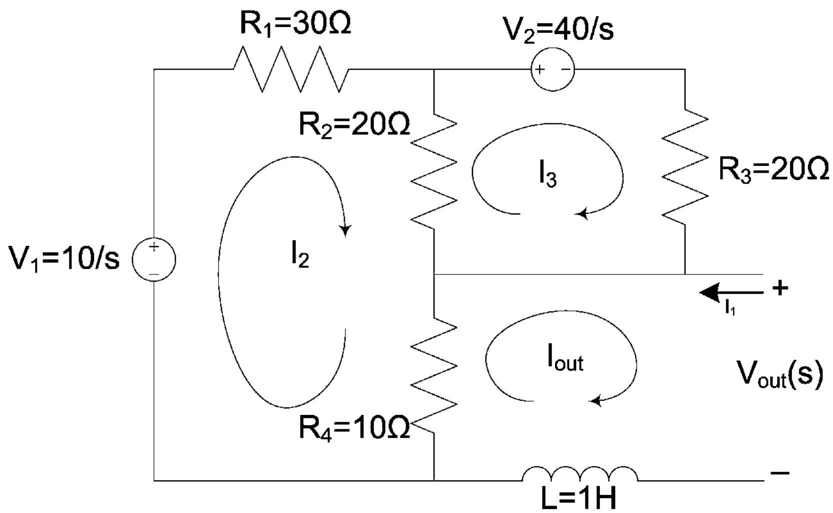
For ω_p and Q_p the subscript p refers to the poles.

$$\begin{aligned}
\omega_p =& \omega_d = \boxed{19.9960} \\
Q_p =& \frac{\omega_p}{2\sigma_p} = \frac{19.9960}{2(0.4)} \approx \boxed{25} \\
\omega_m \approx& \omega_p = \boxed{20[\text{rad/s}]} \\
\omega_{m,exact} =& \omega_p \sqrt{1 - \frac{1}{2Q_p^2}} = (20)\sqrt{1 - \frac{1}{2(25)^2}} = 19.9840[\text{rad/s}] \\
B_w = 2\sigma_p =& \frac{\omega_p}{Q_p} = \boxed{0.8[\text{rad/s}]} \\
\omega_1 \approx& \omega_m - \frac{B_w}{2} = \boxed{19.6[\text{rad/s}]} \\
\omega_2 \approx& \omega_m + \frac{B_w}{2} = \boxed{20.4[\text{rad/s}]}
\end{aligned}$$

The MATLAB code in part(a) produces a figure below:



Problem 72



(a) Thevenin equivalent using Matrix partitioning

Writing loop equations,

$$\begin{aligned} \text{Loop1}: 0 &= -\frac{10}{s} + 30I_2 + 20(I_2 - I_3) + 10(I_2 - I_{out}) \\ \text{Loop2}: 0 &= \frac{40}{s} + 20I_3 + 20(I_3 - I_2) \\ \text{Loop3}: 0 &= V_{out} + sI_{out} + 10(I_{out} - I_2) \end{aligned}$$

$$I_1 = -I_{out}$$

Simplifying and writing into a matrix form,

$$\begin{bmatrix} 10/s \\ 40/s \\ \hline V_{out} \end{bmatrix} = \left[\begin{array}{cc|c} 60 & -20 & -10 \\ 20 & -40 & 0 \\ \hline 10 & 0 & -(s+10) \end{array}\right] \begin{bmatrix} I_2 \\ I_3 \\ \hline I_{out} \end{bmatrix}$$

$$\begin{bmatrix} 10/s \\ 40/s \\ \hline V_{out} \end{bmatrix} = \left[\begin{array}{c|c} W_{11} & W_{12} \\ \hline W_{21} & W_{22} \end{array}\right] \begin{bmatrix} I_2 \\ I_3 \\ \hline I_{out} \end{bmatrix}$$

Writing out the equations,

$$V_{out} = W_{21} \begin{bmatrix} I_2 \\ I_3 \end{bmatrix} + W_{22} I_{out} \tag{eq72.1}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = W_{11} \begin{bmatrix} I_2 \\ I_3 \end{bmatrix} + W_{12} I_{out} \tag{eq72.2}$$

From (eq72.2) we will find the expression for $\begin{bmatrix} I_2 \\ I_3 \end{bmatrix}$ and substitute it back to (eq72.1)

$$\begin{aligned} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} &= W_{11} \begin{bmatrix} I_2 \\ I_3 \end{bmatrix} + W_{12} I_{out} \qquad \text{(eq72.2)} \\ \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - W_{12} I_{out} &= W_{11} \begin{bmatrix} I_2 \\ I_3 \end{bmatrix} \\ W_{11}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - W_{11}^{-1} W_{12} I_{out} &= \begin{bmatrix} I_2 \\ I_3 \end{bmatrix} \end{aligned}$$

$$\Rightarrow \begin{bmatrix} I_2 \\ I_3 \end{bmatrix} = W_{11}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - W_{11}^{-1} W_{12} I_{out} \tag{eq72.3}$$

Now substituting (eq72.3)into (eq.72.1),

$$\begin{aligned} V_{out} &= W_{21} \begin{bmatrix} I_2 \\ I_3 \end{bmatrix} + W_{22} I_{out} \\ V_{out} &= W_{21} \left(W_{11}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - W_{11}^{-1} W_{12} I_{out} \right) + W_{22} I_{out} \\ &= W_{21} W_{11}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - W_{21} W_{11}^{-1} W_{12} I_{out} + W_{22} I_{out} \\ &= W_{21} W_{11}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + (W_{22} - W_{21} W_{11}^{-1} W_{12}) I_{out} \\ &= \underbrace{W_{21} W_{11}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{V_{oc}} + \underbrace{(W_{21} W_{11}^{-1} W_{12} - W_{22})}_{Z_{th}} I_1 \end{aligned}$$

$$
\begin{aligned}
V_{oc} =& W_{21}W_{11}^{-1}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \end{bmatrix}\begin{bmatrix} 60 & -20 \\ 20 & -40 \end{bmatrix}^{-1}\begin{bmatrix} 10/s \\ 40/s \end{bmatrix} \\
=& \begin{bmatrix} 10 & 0 \end{bmatrix}\frac{1}{(60)(-40)-(-20)(20)}\begin{bmatrix} -40 & 20 \\ -20 & 60 \end{bmatrix}\begin{bmatrix} 10/s \\ 40/s \end{bmatrix} \\
=& -\frac{1}{2000}\begin{bmatrix} 10 & 0 \end{bmatrix}\begin{bmatrix} -40 & 20 \\ -20 & 60 \end{bmatrix}\begin{bmatrix} 10/s \\ 40/s \end{bmatrix} \\
=& -\frac{1}{2000}\begin{bmatrix} 10(-40)+0(-20) & 10(20)+0(60) \end{bmatrix}\begin{bmatrix} 10/s \\ 40/s \end{bmatrix} \\
=& -\frac{1}{2000}\begin{bmatrix} -400 & 200 \end{bmatrix}\begin{bmatrix} 10/s \\ 40/s \end{bmatrix} \\
=& -\frac{1}{2000}(-400(10/s)+200(40/s)) \\
=& -\frac{1}{2000}(8000/s) \\
=& -\frac{2}{s}
\end{aligned}
$$

$$
\begin{aligned}
Z_{th} =& W_{21}W_{11}^{-1}W_{12} - W_{22} = \begin{bmatrix} 10 & 0 \end{bmatrix}\begin{bmatrix} 60 & -20 \\ 20 & -40 \end{bmatrix}^{-1}\begin{bmatrix} -10 \\ 0 \end{bmatrix} - \begin{bmatrix} -(s+10) \end{bmatrix} \\
=& -\frac{1}{2000}\begin{bmatrix} 10 & 0 \end{bmatrix}\begin{bmatrix} -40 & 20 \\ -20 & 60 \end{bmatrix}\begin{bmatrix} -10 \\ 0 \end{bmatrix} + (s+10) \\
=& -\frac{1}{2000}\begin{bmatrix} -400 & 200 \end{bmatrix}\begin{bmatrix} -10 \\ 0 \end{bmatrix} + (s+10) \\
=& -\frac{1}{2000}(-4000) + (s+10) \\
=& s+8
\end{aligned}
$$

Using MATLAB to compute V_{oc} and Z_{th},

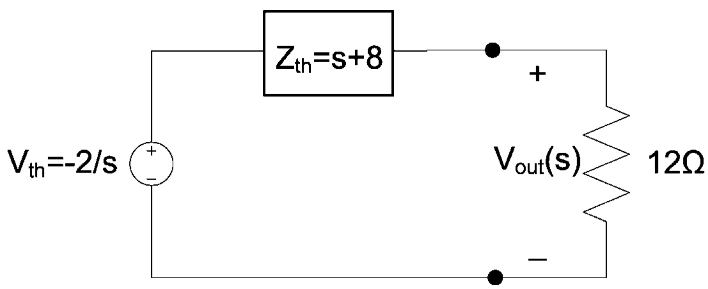
```
syms s I2 I3 Iout V1 V2 Vout Voc Zth
b=[10/s;40/s;Vout];
x=[I2;I3;Iout];
A=[60 -20 -10; 20 -40 0; 10 0 -(s+10)];
W11=[60 -20; 20 -40];
W12=[-10;0];
W21=[10 0];
W22=[-(s+10)];
```

```
Voc=W21*inv(W11)*[10/s;40/s]
% Voc =
%
% -2/s
Zth=W21*inv(W11)*W12-W22
% Zth =
%
% 8+s
```

$$V_{oc} = \boxed{-\frac{2}{s}}$$
$$Z_{th} = \boxed{s+8}$$

(b)

When 12Ω resistor is connected to the output, the circuit becomes:



Simply using voltage division,

$$\begin{aligned} V_{out} &= \frac{12}{(s+8)+12} V_{th} \\ &= \frac{12}{s+20}\left(-\frac{2}{s}\right) \\ &= \frac{-24}{s(s+20)} \end{aligned}$$

Using *ilaplace* command in MATLAB,

$$v_{out}(t) = \boxed{-1.2(1-e^{-20t})u(t)}$$