

ECE 202 LINEAR CIRCUIT ANALYSIS II (SP'09)

Homework #9 Solution (33-36)

Problem 33-34

(a) Find the Laplace transform of $h(t)$

Given

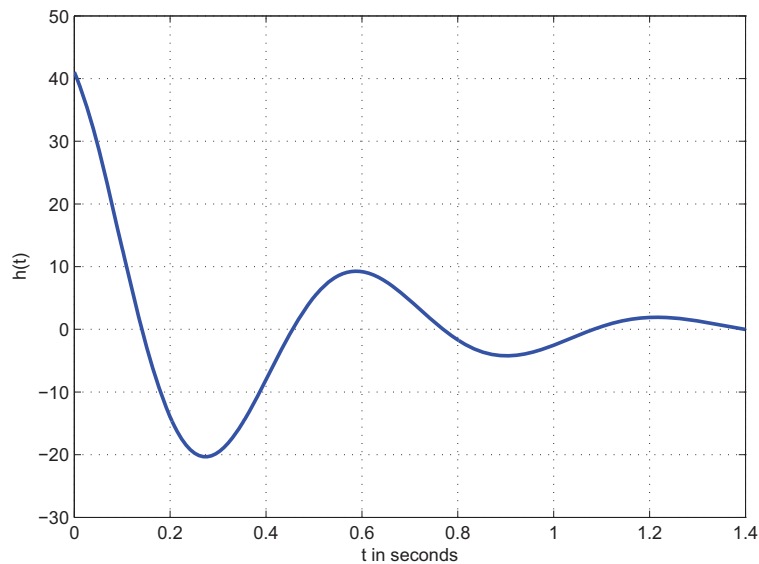
$$\begin{aligned} \text{Impulse Response } h(t) \text{ [V]} & \quad h(t) = Me^{-at} \cos(\omega t - \phi)u(t) \\ \text{Laplace Transform Pair} & \quad \mathcal{L}\{\underbrace{M \cos(\omega t - \phi)u(t)}_{f(t)}\} = M \frac{s \cdot \cos(\phi) + \omega \cdot \sin(\phi)}{s^2 + \omega^2} \\ \text{Freq. Shift Property} & \quad \mathcal{L}\{e^{-at} \cdot f(t)\} = F(s + a) \end{aligned}$$

Setting $M \cos(\omega t - \phi)u(t)$ to $f(t)$, $h(t)$ becomes $h(t) = e^{-at} \cdot f(t)$

Using frequency shift property,

$$\begin{aligned} H(s) = \mathcal{L}\{Me^{-at} \cos(\omega t - \phi)u(t)\} &= M \frac{(\cos \phi)s + (a \cos \phi) + \omega(\sin \phi)}{s^2 + \omega^2} \\ &= \boxed{(M \cos \phi) \frac{s + (a + \omega \tan \phi)}{(s + a)^2 + \omega^2}} \end{aligned}$$

(b) Determine M , ω , and ϕ from the graph and datapoints



datapoint	1	2	3	4	5
t	0	0.141187	0.2738	0.455347	0.587565
$h(t)$	41.0275	0	-20.3319	0	9.27

From the first four datapoints, we get the following equations:

$$41.0275 = M \cos(\phi) \quad (\text{eq1})$$

$$0 = Me^{-at_2} \cos(\omega t_2 - \phi) \quad (\text{eq2})$$

$$-20/3319 = Me^{-at_3} \cos(\omega t_3 - \phi) \quad (\text{eq3})$$

$$0 = Me^{-at_4} \cos(\omega t_4 - \phi) \quad (\text{eq4})$$

In (eq2) and (eq4) cosine terms should be zero to satisfy the condition. From (eq2) and (eq4)

$$\omega t_2 - \phi = \frac{\pi}{2} \quad (\text{eq2.1})$$

$$\omega t_4 - \phi = \frac{3\pi}{2} \quad (\text{eq4.1})$$

Subtracting (eq4.1) from (eq2.1) and simplifying,

$$\omega = \frac{\pi}{(t_4 - t_2)} = \boxed{10[\text{rad/s}]}$$

Now, to find ϕ we use either (eq2.1) or (eq4.1)

$$\phi = \boxed{\omega t_2 - \frac{\pi}{2} = -0.1589[\text{rad/s}]}$$

The values of M can be found using (eq1) and the value of a can be found using (eq3).

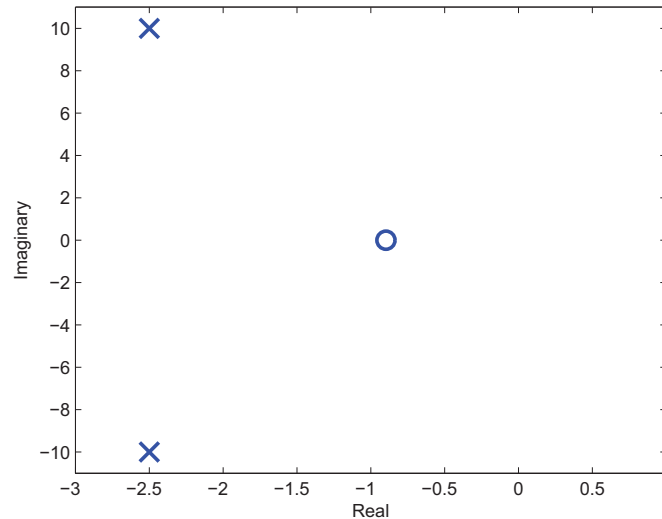
$$\boxed{M=41.5511 \quad a=2.5}$$

(c) Compute $H(s)$ and construct pole zero plot

Using part(a),

$$\boxed{H(s) = 41.0275 \frac{s + 0.8972}{(s + 2.5)^2 + 10^2}}$$

$$z_1 = -0.8972 \quad p_{1,2} = -2.5 \pm j10$$



In the pole-zero plot above, all the poles and zeros lie in the left half complex plane.

- (d) Verify $H_{cir}(s)$ and find R_2 , L , and C given $R_1 = 10\Omega$

$$H_{cir}(s) = \frac{1}{Y_{in}(s)}$$

$$\begin{aligned} Y_{in}(s) &= \frac{1}{R_1} + \frac{1}{Ls + R_2} + sC \\ &= \frac{(sL + R_2) + R_1 + sC(R_1(sL + R_2))}{R_1(sL + R_2)} \\ &= \frac{(R_1LC)s^2 + (R_1R_2C + L)s + (R_1)}{R_1(sL + R_2)} \\ &= \frac{s^2 + \left(\frac{1}{R_1C} + \frac{R_2}{L}\right)s + \left(1 + \frac{R_2}{R_1}\right)\frac{1}{LC}}{\frac{1}{C}\left(s + \frac{R_2}{L}\right)} \end{aligned}$$

$$\therefore H_{cir}(s) = \frac{\frac{1}{C}\left(s + \frac{R_2}{L}\right)}{s^2 + \left(\frac{1}{R_1C} + \frac{R_2}{L}\right)s + \left(1 + \frac{R_2}{R_1}\right)\frac{1}{LC}}$$

To find the values of R_2 , L , and C , we equate $H(s)$ from part (c) to $H_{cir}(s)$ and use the coefficient matching technique.

$$H(s) = 41.0275 \frac{s + 0.8972}{(s^2 + 5s + 106.25)} = \frac{1}{C} \frac{s + \frac{R_2}{L}}{s^2 + \left(\frac{1}{R_1 C} + \frac{R_2}{L}\right)s + \left(1 + \frac{R_2}{R_1}\right) \frac{1}{LC}}$$

Using coefficient matching technique, the values of R_2 , L , and C can be found.

$$\begin{aligned} 41.0275 &= \frac{1}{C} \\ 0.8972 &= \frac{R_2}{L} \\ 5 &= \frac{1}{R_1 C} + \frac{R_2}{L} \\ 106.25 &= \left(1 + \frac{R_2}{R_1}\right) \frac{1}{LC} \end{aligned}$$

$R_1 = 10\Omega$	$R_2 = 0.3589\Omega$	$L = 0.40H$	$C = 0.0244F$
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(e) Step response of the circuit

The step response is the output of the circuit when the input is a step function.

$$\begin{aligned} V_{out} &= \frac{1}{s} H_{cir}(s) \\ &= \frac{1}{s} \cdot 41.0275 \frac{s + 0.8972}{(s^2 + 5s + 106.25)} \end{aligned}$$

Using ilaplace command in MATLAB,

$$\begin{aligned} v_{out-step}(t) &= 0.3465u(t) - (0.3465e^{-2.5t} \cos(10t) + 4.0161e^{-2.5t} \sin(10t)) u(t) \\ v_{out-step}(t) &= \underbrace{0.3465u(t)}_{\text{steadystate}} + \underbrace{1.3968e^{-2.5t} \cos(10t + 1.4847)u(t)}_{\text{transient}} \text{ [V]} \end{aligned}$$

(f) Zero-input response

In s-domain, utilizing a current source combination for the capacitor will be much simpler. For zero-input response, it can be thought as the input current source being replaced by $Cv_c(0-)$, which is due to the initial voltage across the capacitor.

Then, $I_{in}(s) = Cv_c(0-) = 0.0244(10) = 0.244$

so it follows that $V_{out-zeroinput}(s) = 0.244 \cdot H_{cir}(s)$ and $v_{out-zeroinput}(t)$ is simply $0.244h_{cir}(t)$

$$\begin{aligned} v_{out-zeroinput}(t) &= 0.244 \cdot h(t) = Me^{-at} \cos(\omega t - \phi)u(t) \\ &= \boxed{10.1e^{-2.5t} \cos(10t + 0.1589)u(t)} \text{ [V]} \end{aligned}$$

(g) Magnitude and Frequency Scaling

It is given in the question that the new value of $\omega = 20\pi$ so $K_f = 2\pi$ and $K_m = 2$. The new component values are

$$R_{new} = K_m R_{old} = 2(10) = \boxed{20\Omega}$$

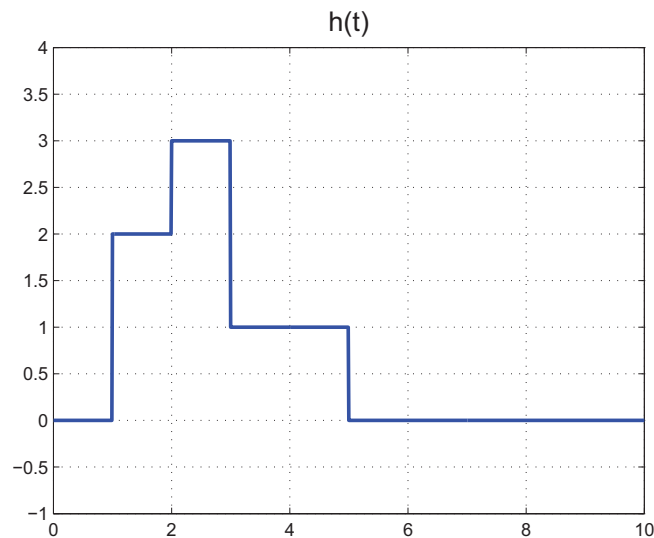
$$L_{new} = \frac{K_m L_{old}}{K_f} = \frac{2(0.4)}{2\pi} = \boxed{\frac{0.4}{\pi}\text{H}}$$

$$C_{new} = \frac{C_{old}}{K_m K_f} = \frac{0.0244}{(2)(20\pi)} = \boxed{\frac{6.1}{\pi}\text{mF}}$$

Therefore, the new transfer function becomes

$$H_{new}(s) = K_m K_f 41.0275 \frac{s + K_f 0.8972}{(s^2 + 5K_f s + 106.25K_f^2)}$$

$$= \boxed{515.57 \frac{s + 5.6373}{(s^2 + 31.416s + 4194.58)}}$$

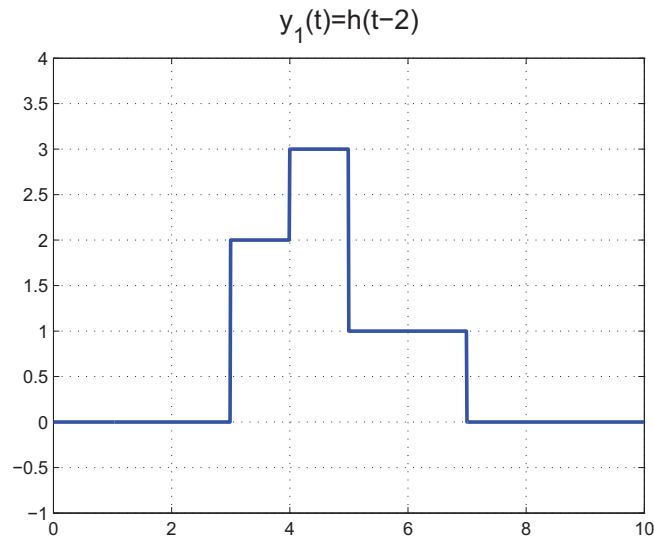
Problem 35

$$h(t) = 2u(t - 1) + u(t - 2) - 2u(t - 3) - u(t - 5)$$

(a) $f_1(t) = \delta(t - 2)$

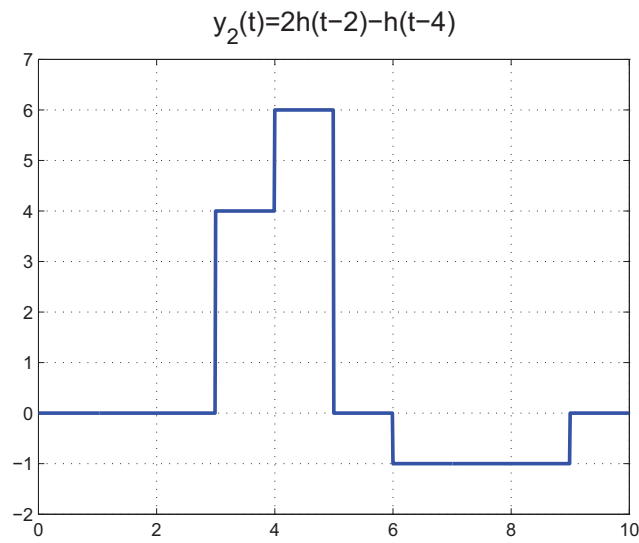
When a function is convoluted with an impulse function, $K\delta(t - T) * f(t) = Kf(t - T)$ that function will be scaled by the strength of the impulse function and shifted by the amount the impulse function is shifted.

$$\begin{aligned}
 y_1(t) &= h(t) * f_1(t) \\
 &= h(t) * \delta(t - 2) \\
 &= h(t - 2) \qquad \qquad \qquad = \boxed{2u(t - 3) + u(t - 4) - 2u(t - 5) - u(t - 7)}
 \end{aligned}$$



(b) $f_2(t) = 2\delta(t - 2) - \delta(t - 4)$

$$\begin{aligned}
 y_2(t) &= h(t) * (2\delta(t - 2) - \delta(t - 4)) \\
 &= 2h(t - 2) - h(t - 4) \\
 &= 2h(t - 2) - h(t - 4) \\
 &= 2((2u(t - 3) + u(t - 4) - 2u(t - 5) - u(t - 7)) - (2u(t - 5) + u(t - 6) - 2u(t - 7) - u(t - 9))) \\
 &= \boxed{4u(t - 3) + 2u(t - 4) - 6u(t - 5) - u(t - 6) + u(t - 9)}
 \end{aligned}$$



(c) $f_3(t) = 2u(t)$

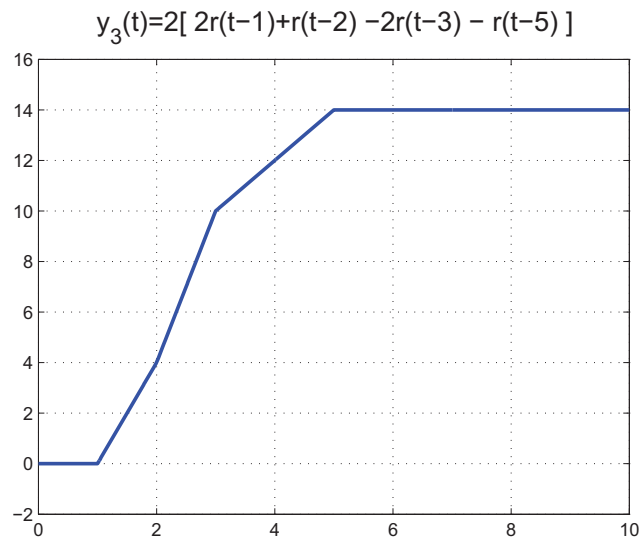
$$u(t) * u(t) = r(t) \tag{35c.1}$$

$$f(t) * h(t) = y(t) \tag{35c.2}$$

$$f(t) * h(t - T) = f(t - T) * h(t) = y(t - T) \tag{35c.3}$$

$h(t)$ expressed as a sum of scaled/shifted step functions is as follows: $h(t) = 2u(t - 1) + u(t - 2) - 2u(t - 3) - u(t - 5)$ Using the property 35c.3,

$$y_3(t) = 4r(t - 1) + 2r(t - 2) - 4r(t - 3) - 2r(t - 5)$$



Problem 36

$$f(t) = Ke^{at}u(-t), t < 0, a > 0$$

(a) $y_1(t)$

$$H_1(s) = Ke^{-2s} \implies h_1(t) = K\delta(t - 2)$$

$$\begin{aligned} y_1(t) &= f(t) * h_1(t) \\ &= Ke^{at}u(-t) * K\delta(t - 2) \\ &= \boxed{K^2 e^{a(t-2)} u(2-t)} \end{aligned}$$

(b) $y_2(t)$

$$H_2(s) = \frac{K}{s} \implies h_2(t) = Ku(t)$$

$$\begin{aligned}
 y_2(t) &= f(t) * h_2(t) \\
 &= Ke^{at}u(-t) * Ku(t) \\
 &= \int_{-\infty}^{\infty} Ke^{a\tau} \underbrace{u(-\tau)}_{\begin{cases} 1 & \tau \leq 0 \\ 0 & \tau > 0 \end{cases}} Ku(t-\tau)d\tau \\
 &= K^2 \int_{-\infty}^0 e^{a\tau} \underbrace{u(t-\tau)}_{\begin{cases} 1 & \tau \leq t \\ 0 & \tau > t \end{cases}} d\tau
 \end{aligned}$$

Case1: $t < 0$

Case2: $t > 0$

$$\begin{aligned}
 y_2(t) &= K^2 \int_{-\infty}^t e^{a\tau} d\tau \\
 &= \frac{K^2}{a} \left[e^{a\tau} \Big|_{-\infty}^t \right] \\
 &= \frac{K^2}{a} e^{at}
 \end{aligned}$$

$$\begin{aligned}
 y_2(t) &= K^2 \int_{-\infty}^0 e^{a\tau} d\tau \\
 &= \frac{K^2}{a} \left[e^{a\tau} \Big|_{-\infty}^0 \right] \\
 &= \frac{K^2}{a}
 \end{aligned}$$

$$y_2(t) = \begin{cases} \frac{K^2}{a} e^{at} & t \leq 0 \\ \frac{K^2}{a} & t > 0 \end{cases} = \boxed{\frac{K^2}{a} [e^{at}u(-t) + u(t)]}$$

(c) $y_3(t)$

$$\begin{aligned}
 H_3(s) &= \frac{K}{s+a}(1 - e^{-2a-2s}) = \frac{K}{s+a} + \frac{-Ke^{-2a-2s}}{(s+a)} \\
 h_3(t) &= \underbrace{Ke^{-at}u(t)}_{h_{3a}(t)} - \underbrace{Ke^{-2a}e^{-a(t-2)}u(t-2)}_{h_{3b}(t)} \\
 &\text{and } h_{3b}(t) = -(e^{-2a})h_{3a}(t-2)
 \end{aligned}$$

So, I will find the response due to $h_{3a}(t)$ and use linearity and time shift property to find the total response.

$$\begin{aligned}
 y_{3a}(t) &= f(t) * h_{3a}(t) \\
 &= K e^{at} u(-t) * K e^{-at} u(t) \\
 &= K^2 \int_{-\infty}^{\infty} e^{a\tau} \underbrace{u(-\tau)}_{\begin{cases} 1 & \tau \leq 0 \\ 0 & \tau > 0 \end{cases}} e^{-a(t-\tau)} u(t-\tau) d\tau \\
 &= K^2 \int_{-\infty}^0 e^{a\tau} e^{-a(t-\tau)} \underbrace{u(t-\tau)}_{\begin{cases} 1 & \tau \leq t \\ 0 & \tau > t \end{cases}} d\tau
 \end{aligned}$$

Case1: $t < 0$

$$\begin{aligned}
 y_{3a}(t) &= K^2 e^{-at} \int_{-\infty}^t e^{2a\tau} d\tau \\
 &= \frac{K^2 e^{-at}}{2a} \left[e^{2a\tau} \Big|_{-\infty}^t \right] \\
 &= \frac{K^2}{2a} e^{at}
 \end{aligned}$$

Case2: $t > 0$

$$\begin{aligned}
 y_{3a}(t) &= K^2 e^{-at} \int_{-\infty}^0 e^{2a\tau} d\tau \\
 &= \frac{K^2 e^{-at}}{2a} \left[e^{2a\tau} \Big|_{-\infty}^0 \right] \\
 &= \frac{K^2}{2a} e^{-at}
 \end{aligned}$$

$$y_{3a}(t) = \begin{cases} \frac{K^2}{2a} e^{at} & t \leq 0 \\ \frac{K^2}{2a} e^{-at} & t > 0 \end{cases} = \boxed{\frac{K^2}{2a} [e^{at} u(-t) + e^{-at} u(t)]}$$

Using linearity and time shift property,

$$\begin{aligned}
 y_{3b}(t) &= -(e^{-2a}) y_{3a}(t-2) \\
 &= -(e^{-2a}) \frac{K^2}{2a} \left[e^{a(t-2)} u(-(t-2)) + e^{-a(t-2)} u(t-2) \right] \\
 &= \boxed{-\frac{K^2}{2a} [e^{a(t-4)} u(2-t) + e^{-at} u(t-2)]}
 \end{aligned}$$

Now, we use Linearity property to get the total response, $y_3(t)$

$$\begin{aligned}
 y_3(t) &= y_{3a}(t) + y_{3b}(t) \\
 &= \boxed{\frac{K^2}{2a} \left(e^{at} \left(u(-t) - e^{-4a} u(2-t) \right) + e^{-at} \left(u(t) - u(t-2) \right) \right)}
 \end{aligned}$$