# EE648 (CC761-M) DSP II

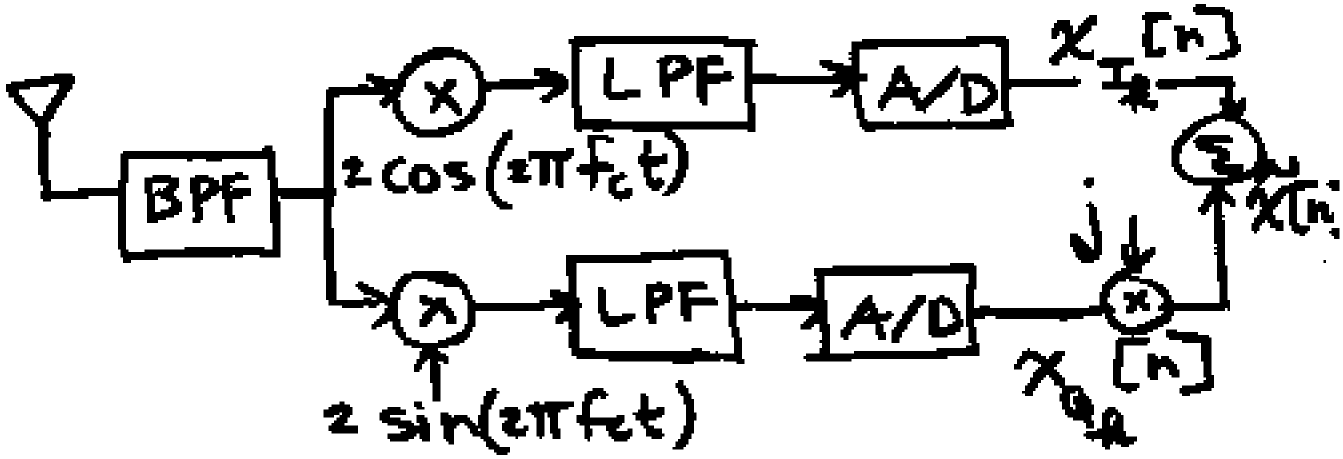
## Session 10 (Live: 2/11/99)

Outline

- Space-Time Signal Processing
  - Narrowband model for a linear array of antennas

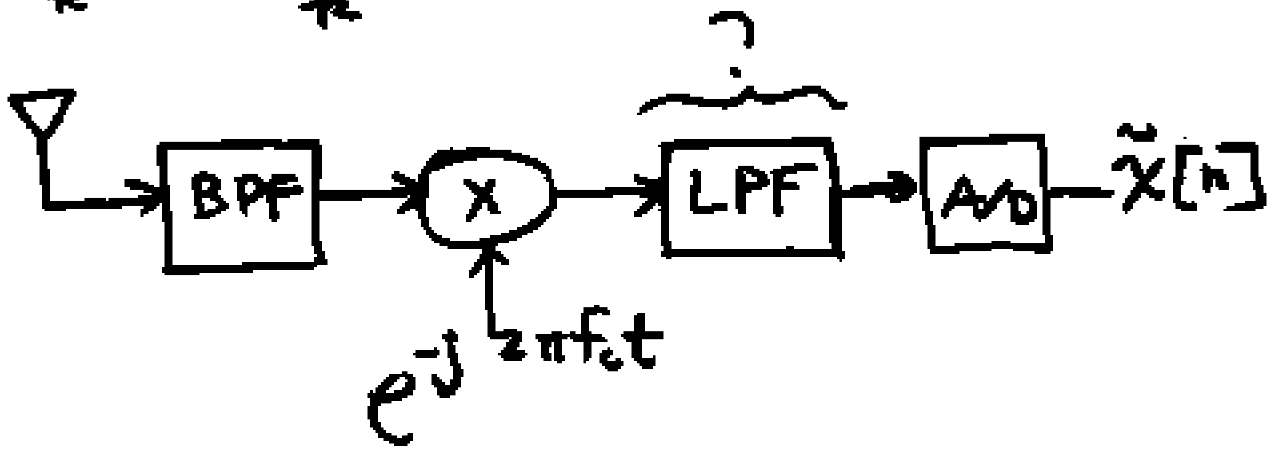
- Significance of complex-envelope representation

$$s_R(t) = s_{I_R}(t)\cos(2\pi f_c t) - s_{Q_R}(t)\sin(2\pi f_c t)$$



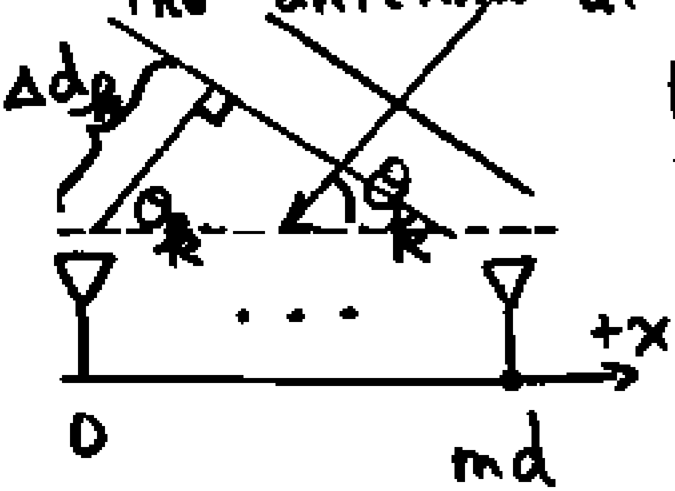
- same result with:

$$\hat{s}_R(t) = \tilde{s}_R(t)e^{j2\pi f_c t}$$




- where: $\tilde{s}_R(t) = s_{I_R}(t) + j\,s_{Q_R}(t)$
- development to follow uses complex envelope representation

• consider the antenna at $x=0$ and the antenna at $x = md$




propagation delay between the two antennas :

$$\tau_{m_k} = \frac{md\cos\theta_k}{c}$$

$\Delta d_k = md\cos\theta_k$

since: $\cos\theta_k = \dfrac{\Delta d_k}{md}$

$c$ = speed of light
= speed of E-M waves in free space

$\tilde{x}(m;t)$ = output of m-th antenna using complex envelope representation

$$\tilde{x}(m;t) = x(0;t+\tau_{m_k})$$

$$= \tilde{s}_k(t+\tau_{m_k})\, e^{j2\pi f_c(t+\tau_{m_k})}$$

$$= \tilde{s}_k(t+\tau_{m_k})\, e^{j2\pi f_c \tau_{m_k}}\, e^{j2\pi f_c t}$$

$$= \tilde{s}_k\left(t+\frac{md\cos\theta_k}{c}\right) e^{j\frac{2\pi f_c m d\cos\theta_k}{c}} \cdot e^{j2\pi f_c t}$$

- note: $c = f_c \lambda_c$ ← wavelength
- define:

$$x(m;t) = \tilde{x}(m;t)\, e^{-j 2\pi f_c t}$$

$$= \tilde{s}_k\left(t + \frac{m d \cos\theta_k}{c}\right) e^{j \frac{2\pi m}{\lambda_c} d \cos\theta_k}$$

- narrowband assumption:

$$\tilde{s}_k\left(t + \frac{m d \cos\theta_k}{c}\right) \approx \tilde{s}_k(t)$$

• justification: use practical nos. related to cellular communications

• $\max\{md\} \approx 10\lambda_c$ $\left(\begin{array}{l}10 \text{ ft. at} \\ f_c = 1 \text{ GHz}\end{array}\right)$

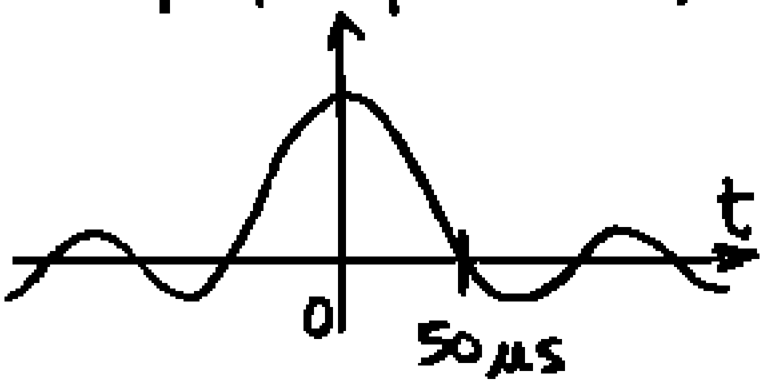
• $\max\{\cos\theta_k\} = 1$

• $c = 3\times10^8 \frac{m}{s} = f_c\lambda_c = 10^9\lambda_c$

• $\max\{\tau_{m_k}\} = \max\left\{\frac{md\cos\theta_k}{c}\right\}$

$\approx \frac{10\lambda_c}{10^9\lambda_c} = 10^{-8}\,s = 10 \text{ ns}$

• IS-136 TDMA Cellular Standard employs pulse symbol waveform




50 μs vs. 10 ns. $\Rightarrow$ negligible propagation delay

• IS-95 CDMA Cellular Standard employs ~1 μs "chip" waveform

1 μs = 1000 ns. vs. 10 ns

$\Rightarrow$ still reasonable approx.

- define spatial frequency:

$$\mu_k = \frac{2\pi}{\lambda_c} d \cos\theta_k$$

- $x(m;t) = \tilde{s}_k(t)\, e^{j\mu_k m}$

- with $P$ co-channel signals, linearity / superposition dictates

$$x(m;t) = \sum_{k=1}^{P} \tilde{s}_k(t)\, e^{j\mu_k m}$$

- $\mu_k = \frac{2\pi}{\lambda_c} d \cos\theta_k$
- consider $d = \frac{\lambda_c}{2}$ = half-wavelength separation between adjacent antennas

$$\mu_k = \pi \cos\theta_k$$

- 1-to-1 mapping between

$$0^\circ < \theta_k < 180^\circ \underset{\text{1-to-1}}{\Longleftrightarrow} -\pi < \mu_k < \pi$$

- Consider $d > \frac{\lambda_c}{2}$ :

$$e^{j\frac{2\pi}{\lambda_c} d \cos\theta_k} = e^{j\frac{2\pi}{\lambda_c} d \cos\theta_k^{(a)}}$$

- can only occur if:

$$\frac{2\pi}{\lambda_c} d \cos\theta_k^{(a)} = \frac{2\pi}{\lambda_c} d \cos\theta_k + \ell 2\pi$$

$\ell$, integer

$$\cos\theta_k^{(a)} = \cos\theta_k + \ell \frac{\lambda_c}{d}$$

$$\theta_k^{(a)} = \cos^{-1}\left\{\cos\theta_k + \ell \frac{\lambda_c}{d}\right\}$$

- if $d < \frac{\lambda_c}{2} \Rightarrow \frac{\lambda_c}{d} > 2$

and no sol'n $\Rightarrow$ no ambiguity

- but, if $d > \frac{\lambda_c}{2}$, then there exists ambiguous angles

- Nyquist spatial sampling rate:

$$d < \frac{\lambda_c}{2} \Rightarrow \frac{1}{d} > \frac{2}{\lambda_c}$$

- at this point:

$$x(m,t) = \sum_{k=1}^{P} \tilde{s}_k(t) e^{j \mu_k m}$$

$m = 0, 1, \ldots, M-1 \Rightarrow M$ antennas

- sampling in time "behind" each antenna:

$$x[m,n] = \tilde{x}(m, nT_s)$$

$$= \sum_{k=1}^{P} \tilde{s}_k[n] e^{j m \mu_k} \quad n = 0, 1, \ldots, N-1$$

- for fixed DT $n = n_0$;
$x[m; n_0]$ is a sum of complex sinewaves as a fn. of the discrete-space index $m$
- apply any of the parametric spectral estimation techniques to estimate $\mu_k$ and, hence, $\theta_k$, $k = 1, \ldots, P$

- define spatial correlations as

$$\phi[m,\ell] = E\{ x[m;n]\, x^*[\ell;n] \}$$

- time-avg'd estimate of the spatial correlation between the m-th and $\ell$-th antennas

$$\hat{\phi}[m,\ell] = \frac{1}{N} \sum_{n=0}^{N-1} x[m;n]\, x^*[\ell;n]$$

- if the incident signals, $\tilde{s}_k(t)$, $k=1,\ldots,P$ are independent zero-mean stationary random processes, then

$$E\{x[m;n]\,x^*[\ell;n]\} = r[m-\ell]$$

PROOF:

$$E\left\{\sum_{k=1}^{P} \tilde{s}_k[n]e^{jm\mu_k}\sum_{q=1}^{P}\tilde{s}_q^*[n]e^{-j\ell\mu_q}\right\}$$

$$=\sum_{k=1}^{P}\sum_{q=1}^{P} E\{\tilde{s}_k[n]\,\tilde{s}_q^*[n]\}e^{j(m\mu_k-\ell\mu_q)}$$

- since $\tilde{s}_k[n]$ and $\tilde{s}_q[n]$ are independent for $k \neq q$

$$E\{\tilde{s}_k[n]\,\tilde{s}_q^*[n]\}$$
$$= E\{\tilde{s}_k[n]\}\,E\{\tilde{s}_q^*[n]\}$$
$$= 0 \cdot 0 \quad \text{(zero mean)}$$

- for $k = q$ : $E\{|\tilde{s}_k[n]|^2\} = \sigma_k^2$
- Thus:

$$E\{\tilde{s}_k[n]\,\tilde{s}_q^*[n]\} = \sigma_k^2\,\delta[k-q]$$

- thus, double sum reduces to single sum:

$$E\{x[m;n]\,x^*[\ell;n]\} = \sum_{k=1}^{P} \sigma_k^2 e^{j(m-\ell)\mu_k} = r[m-\ell]$$

$\Rightarrow$ spatially wide-sense stationary