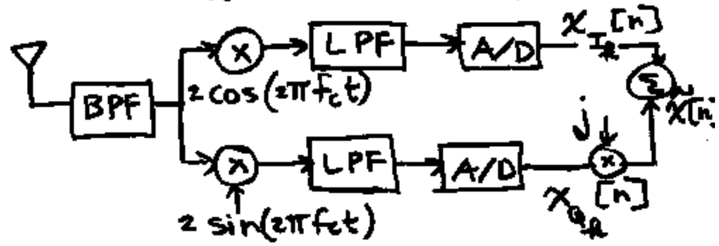
EE648 (CC761-M) DSP II
Session 10 (Live: 2/11/99)

<u>Outline</u>

- Space-Time Signal Processing
  - Narrowband model for a linear array of antennas

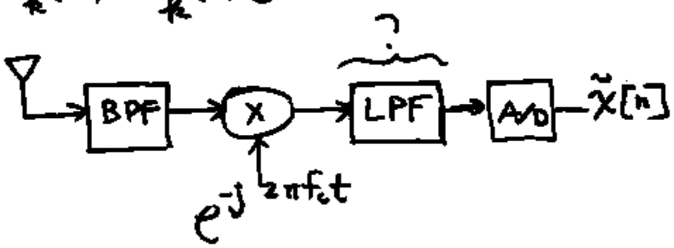
- Significance of complex-envelope representation

$$s_k(t) = s_{I_k}(t)\cos(2\pi f_c t) - s_{Q_k}(t)\sin(2\pi f_c t)$$



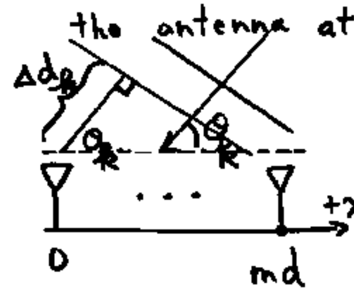
- same result with:

$$\hat{s}_k(t) = \tilde{s}_k(t)e^{j2\pi f_c t}$$



- where: $\tilde{s}_k(t) = s_{I_k}(t) + j\,s_{Q_k}(t)$
- development to follow uses complex envelope representation

- consider the antenna at $x=0$ and the antenna at $x = md$



propagation delay between the two antennas:

$$\tau_{m_k} = \frac{md\cos\theta_k}{c}$$

c = speed of light = speed of E-M waves in free space

$$\Delta d_k = md\cos\theta_k$$

since: $\cos\theta_k = \dfrac{\Delta d_k}{md}$

$\tilde{x}(m;t)$ = output of m-th antenna using complex envelope representation

$$\tilde{x}(m;t) = x(0;t+\tau_{m_k})$$

$$= \tilde{s}_k(t+\tau_{m_k})\, e^{j2\pi f_c (t+\tau_{m_k})}$$

$$= \tilde{s}_k(t+\tau_{m_k})\, e^{j2\pi f_c \tau_{m_k}}\, e^{j2\pi f_c t}$$

$$= \tilde{s}_k\left(t+\frac{md\cos\theta_k}{c}\right) e^{j\frac{2\pi f_c m d \cos\theta_k}{c}} \cdot e^{j2\pi f_c t}$$

• note: $c = f_c \lambda_c$ ← wavelength

• define:

$$x(m;t) = \tilde{x}(m;t)\, e^{-j2\pi f_c t}$$

$$= \tilde{s}_k\left(t+\frac{md\cos\theta_k}{c}\right) e^{j\frac{2\pi m}{\lambda_c} d\cos\theta_k}$$

• narrowband assumption:

$$\tilde{s}_k\left(t+\frac{md\cos\theta_k}{c}\right) \approx \tilde{s}_k(t)$$

• justification: use practical nos. related to cellular communications

• $\max\{md\} \approx 10\lambda_c$ (10 ft. at $f_c = 1$ GHz)

• $\max\{\cos\theta_k\} = 1$

• $c = 3\times 10^8 \frac{m}{s} = f_c \lambda_c = 10^9 \lambda_c$

• $\max\{\tau_{m_k}\} = \max\left\{\frac{md\cos\theta_k}{c}\right\}$

$$\approx \frac{10\lambda_c}{10^9\lambda_c} = 10^{-8}\, s = 10\ ns$$

• IS-135 TDMA Cellular Standard employs pulse symbol waveform

50 μs vs. 10 ns. ⇒ negligible propagation delay

• IS-95 CDMA Cellular Standard employs ~1μs "chip" waveform

1μs = 1000 ns. vs. 10 ns

⇒ still reasonable approx.

• define spatial frequency:

$$\mu_k = \frac{2\pi}{\lambda_c} d \cos\theta_k$$

• $x(m;t) = \tilde{s}_k(t)\, e^{j\mu_k m}$

• with P co-channel signals, linearity / superposition dictates

$$x(m;t) = \sum_{k=1}^{P} \tilde{s}_k(t)\, e^{j\mu_k m}$$

• $\mu_k = \frac{2\pi}{\lambda_c} d \cos\theta_k$

• consider $d = \frac{\lambda_c}{2}$ = half-wavelength separation between adjacent antennas

$$\mu_k = \pi \cos\theta_k$$

• 1-to-1 mapping between

$$0^\circ < \theta_k < 180^\circ \underset{1\text{-to-}1}{\Longleftrightarrow} -\pi < \mu_k < \pi$$

• Consider $d > \frac{\lambda_c}{2}$:

$$e^{j\frac{2\pi}{\lambda_c} d \cos\theta_k} = e^{j\frac{2\pi}{\lambda_c} d \cos\theta_k^{(a)}}$$

• can only occur if:

$$\frac{2\pi}{\lambda_c} d \cos\theta_k^{(a)} = \frac{2\pi}{\lambda_c} d \cos\theta_k + \ell 2\pi \qquad \ell, \text{integer}$$

$$\cos\theta_k^{(a)} = \cos\theta_k + \ell\frac{\lambda_c}{d}$$

$$\theta_k^{(a)} = \cos^{-1}\left\{\cos\theta_k + \ell\frac{\lambda_c}{d}\right\}$$

• if $d < \frac{\lambda_c}{2} \Rightarrow \frac{\lambda_c}{d} > 2$

and no sol'n $\Rightarrow$ no ambiguity

• but, if $d > \frac{\lambda_c}{2}$, then there exists ambiguous angles

• Nyquist spatial sampling rate:

$$d < \frac{\lambda_c}{2} \Rightarrow \frac{1}{d} > \frac{2}{\lambda_c}$$

- at this point:

$$x(m;t) = \sum_{k=1}^{P} \tilde{s}_k(t) e^{j\mu_k m}$$

$m = 0, 1, \ldots, M-1 \Rightarrow M$ antennas

- sampling in time "behind" each antenna:

$$x[m;n] = \tilde{x}(m; nT_s)$$
$$= \sum_{k=1}^{P} \tilde{s}_k[n] e^{jm\mu_k} \quad n = 0, 1, \ldots, N-1$$

- for fixed DT $n = n_0$;

$x[m; n_0]$ is a sum of complex sinewaves as a fn. of the discrete-space index m

- apply any of the parametric spectral estimation techniques to estimate μ_k and, hence, θ_k, $k = 1, \ldots, P$

- define spatial correlations as

$$\phi[m,\ell] = E\{x[m;n]\, x^*[\ell;n]\}$$

- time-avg'd estimate of the spatial correlation between the m-th and ℓ-th antennas

$$\hat{\phi}[m,\ell] = \frac{1}{N}\sum_{n=0}^{N-1} x[m;n]\, x^*[\ell;n]$$

- if the incident signals, $\tilde{s}_k(t)$, $k = 1, \ldots, P$ are independent zero-mean stationary random processes, then

$$E\{x[m;n]\, x^*[\ell;n]\} = r[m-\ell]$$

PROOF:

$$E\left\{\sum_{k=1}^{P} \tilde{s}_k[n] e^{jm\mu_k} \sum_{q=1}^{P} \tilde{s}_q^*[n] e^{-j\ell\mu_q}\right\}$$
$$= \sum_{k=1}^{P}\sum_{q=1}^{P} E\{\tilde{s}_k[n]\, \tilde{s}_q^*[n]\} e^{j(m\mu_k - \ell\mu_q)}$$

- since $\tilde{s}_k[n]$ and $\tilde{s}_q[n]$ are independent for $k \neq q$

$$E\{\tilde{s}_k[n]\,\tilde{s}_q^*[n]\}$$
$$= E\{\tilde{s}_k[n]\}\,E\{\tilde{s}_q^*[n]\}$$
$$= 0 \cdot 0 \qquad \text{(zero mean)}$$

- for $k=q$: $E\{|\tilde{s}_k[n]|^2\} = \sigma_k^2$
- Thus:

$$E\{\tilde{s}_k[n]\,\tilde{s}_q^*[n]\} = \sigma_k^2\,\delta[k-q]$$

- thus, double sum reduces to single sum:

$$E\{x[m;n]\,x^*[\ell;n]\}$$
$$= \sum_{k=1}^{P} \sigma_k^2\, e^{j(m-\ell)\mu_k} = r[m-\ell]$$

=> spatially wide-sense stationary