EE648 (CC761-M) DSP II

Session 11 (live: 2/16/99)

Outline

- FIR Spatial Filters-Beamforming
- Spatial Spectrum Estimation

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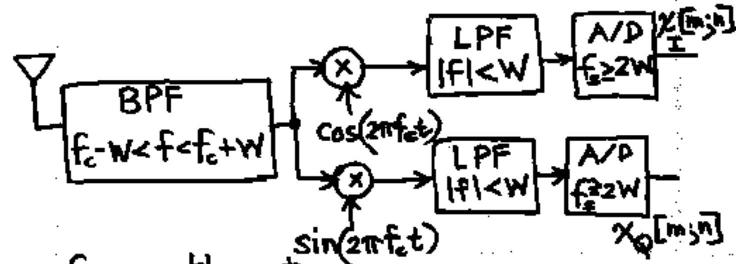
- basic module "behind" each antenna:



- for m-th antenna:
  $\tilde{x}[m;n] = x_I[m;n] + j\,x_Q[m;n]$
- drop tilde for notational simplicity
  - show multiple antennas

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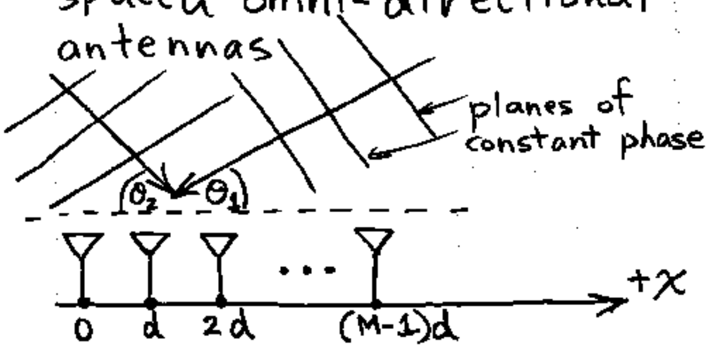
- Consider special case of linear array of equi-spaced omni-directional antennas



- model developed previously in the case of P co-channel planewaves (narrowband) incident upon a linear array of M antennas

$$x[m;n] = \sum_{k=1}^{P} s_k[n]\, e^{j m \mu_k} \qquad m = 0, 1, \ldots, M-1$$

$$\mu_k = \frac{2\pi}{\lambda} d \cos\theta_k \qquad k = 1, \ldots, P$$

• define $M \times 1$ spatial "snapshot" vector at time n:

$$\underline{x}[n] = \left[x[0;n], x[1;n], \ldots, x[M-1;n]\right]^T$$

$$= \sum_{k=1}^{P} s_k[n]\, \underline{s}(\mu_k) + \underline{\nu}[n]$$

• where:

$$\underline{s}(\mu_k) = \left[1, e^{j\mu_k}, e^{j2\mu_k}, \ldots, e^{j(M-1)\mu_k}\right]^T$$

• and additive noise

$$\underline{\nu}[n] = \left[\nu[0;n], \nu[1;n], \ldots, \nu[M-1;n]\right]^T$$

• recall: FIR filter in temporal domain:

$$y[n] = \sum_{k=0}^{M-1} h[k]\, x[n-k]$$

• in vector form:

$$y[n] = \underline{h}_M^T\, \underline{x}[n]$$

• where:

$$\underline{h}_M = \left[h[0], h[1], \ldots, h[M-1]\right]^T$$

$$\underline{x}[n] = \left[x[n], x[n-1], \ldots, x[n-(M-1)]\right]^T$$

• similarly, spatial FIR filtering is effected as

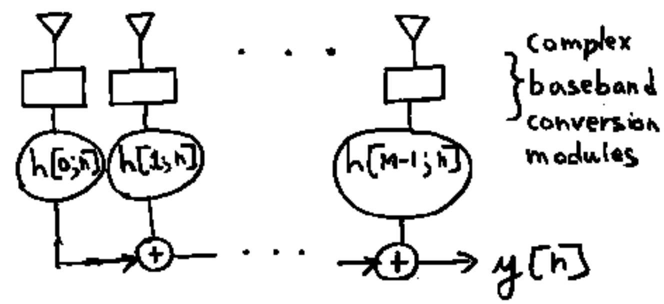
$$y[n] = \sum_{m=0}^{M-1} h^*[m;n]\, x[m;n]$$

• in vector form: $y[n] = \underline{h}_M^H[n]\, \underline{x}[n]$

$H = *T$ (Hermitian Transpose)

$$\underline{h}_M[n] = \left[h[0;n], h[1;n], \ldots, h[M-1;n]\right]^T$$

$$\underline{x}[n] = \left[x[0;n], x[1;n], \ldots, x[M-1;n]\right]^T$$



- consider for analysis purposes spatial FIR filter is Time-Invariant
- consider effect with P co-channel planewaves plus noise:

$$y[n] = \underline{h}_M^H \underline{x}[n]$$

$$= \sum_{k=1}^{P} s_k[n]\left(\underline{h}_M^H \underline{s}(\mu_k)\right) + \underline{h}_M^H \underline{\nu}[n]$$

- Consider WLOG enumeration is such that $k=1$ corresponds to "desired signal"

$$y[n] = s_1[n]\left(\underline{h}_M^H \underline{s}(\mu_1)\right)$$

$$+ \sum_{k=2}^{P} s_k[n]\left(\underline{h}_M^H \underline{s}(\mu_k)\right) \quad \text{interference contribution}$$

$$+ \underline{h}_M^H \underline{\nu}[n] \quad \text{noise contribution}$$

$$\mu_k = \frac{2\pi}{\lambda_c} d \cos\theta_k \qquad k = 1, \ldots, P$$

- Minimum Variance Spatial Filtering (Beamforming)
- criterion for designing FIR Spatial Filter for a specific angle θ_1 (μ_1):
- minimize the mean square value of the spatial filter output subject to the constraint that a planewave from direction θ_1 (μ_1) be "passed" with unity gain

- Mathematical Formulation:

$$\min_{\underline{h}_M} E\{|y[n]|^2\}$$

subject to: $\underline{h}_M^H \underline{s}(\mu_1) = 1$

- Objective function:

$$E\{|y[n]|^2\} = E\{\underline{h}_M^H \underline{x}[n]\, \underline{h}_M^T \underline{x}^*[n]\}$$

$$= E\{\underline{h}_M^H \underline{x}[n]\, \underline{x}^H[n]\, \underline{h}_M\}$$

$$E\{|y[n]|^2\} = \underline{h}_M^H \underline{R}_{xx} \underline{h}_M$$

- where: $\underline{R}_{xx} = E\{\underline{x}[n]\,\underline{x}^H[n]\}$
- $M \times M$ spatial correlation matrix

$$R_{xx}[m,\ell] = E\{x[m;n]\,x^*[\ell;n]\}$$

- in practice, $\underline{R}_{xx}$ is estimated via a time-average: $\hat{\underline{R}}_{xx} = \frac{1}{N}\sum_{n=0}^{N-1} \underline{x}[n]\underline{x}^H[n]$

$$\min_{\underline{h}_M} \underline{h}_M^H \underline{R}_{xx} \underline{h}_M \quad \text{s.t.} \quad \underline{h}_M^H \underline{s}(\mu_1) = 1$$

- Solve via method of Lagrange multipliers - converts constrained optimization to unconstrained one

$$\min_{\underline{h}_M} \underline{h}_M^H \underline{R}_{xx} \underline{h}_M + \lambda[1 - \underline{h}_M^H \underline{s}(\mu_1)]$$

- Recall:

$$\nabla_{\underline{h}_M}(\underline{h}_M^H \underline{R}_{xx} \underline{h}_M) = 2\underline{R}_{xx} \underline{h}_M$$

$$\nabla_{\underline{h}_M}(\underline{h}_M^H \underline{s}(\mu_1)) = \underline{s}(\mu_1)$$

- Taking gradient of augmented function using these results:

$$2\underline{R}_{xx}\underline{h}_M - \lambda \underline{s}(\mu_1) = 0$$

$$\underline{h}_M = \frac{\lambda}{2}\underline{R}_{xx}^{-1}\underline{s}(\mu_1)$$

- λ is determined by satisfying the constraint

$$\underline{h}_M^H \underline{s}(\mu_1) = \left(\frac{\lambda^*}{2}\underline{s}^H(\mu_1)\underline{R}_{xx}^{-1}\right)\underline{s}(\mu_1) = 1$$

$$\lambda^* = \frac{2}{\underline{s}^H(\mu_1)\underline{R}_{xx}^{-1}\underline{s}(\mu_1)}$$

- note: if $\underline{A}^H = \underline{A}$, $\underline{x}^H\underline{A}\underline{x}$ is real-valued

$$\underline{h}_M^{opt} = \frac{\underline{R}_{xx}^{-1}\underline{s}(\mu_1)}{\underline{s}^H(\mu_1)\underline{R}_{xx}^{-1}\underline{s}(\mu_1)}$$

- mean square value of Spatial FIR Filter output using optimum filter: $E\{|y[n]|^2\} = \underline{h}_M^{opt\,H} \underline{R}_{xx} \underline{h}_M^{opt}$

$$= \frac{\left(\underline{s}^H(\mu_1)\underline{R}_{xx}^{-1}\right)\underline{R}_{xx}\left(\underline{R}_{xx}^{-1}\underline{s}(\mu_1)\right)}{\left(\underline{s}^H(\mu_1)\underline{R}_{xx}^{-1}\underline{s}(\mu_1)\right)^2}$$

$$= \frac{1}{\underline{s}^H(\mu_1)\underline{R}_{xx}^{-1}\underline{s}(\mu_1)}$$

$$\hat{S}_{xx}^{MV}(\mu) = \frac{1}{\underline{s}^H(\mu)\underline{R}_{xx}^{-1}\underline{s}(\mu)}$$

- for any μ, the spatial spectrum $\hat{S}_{xx}^{MV}(\mu)$ is ideally an estimate of the energy arriving from the direction θ assoc. with $\mu = \frac{2\pi}{\lambda_c} d \cos\theta$