EE648 (CC761-M) DSP II

Session 15 (Live: 3/2/99)

Outline:

- Eigenstructure Techniques for Spatial Spectrum Estimation
- MUSIC: MUltiple SIgnal Classification

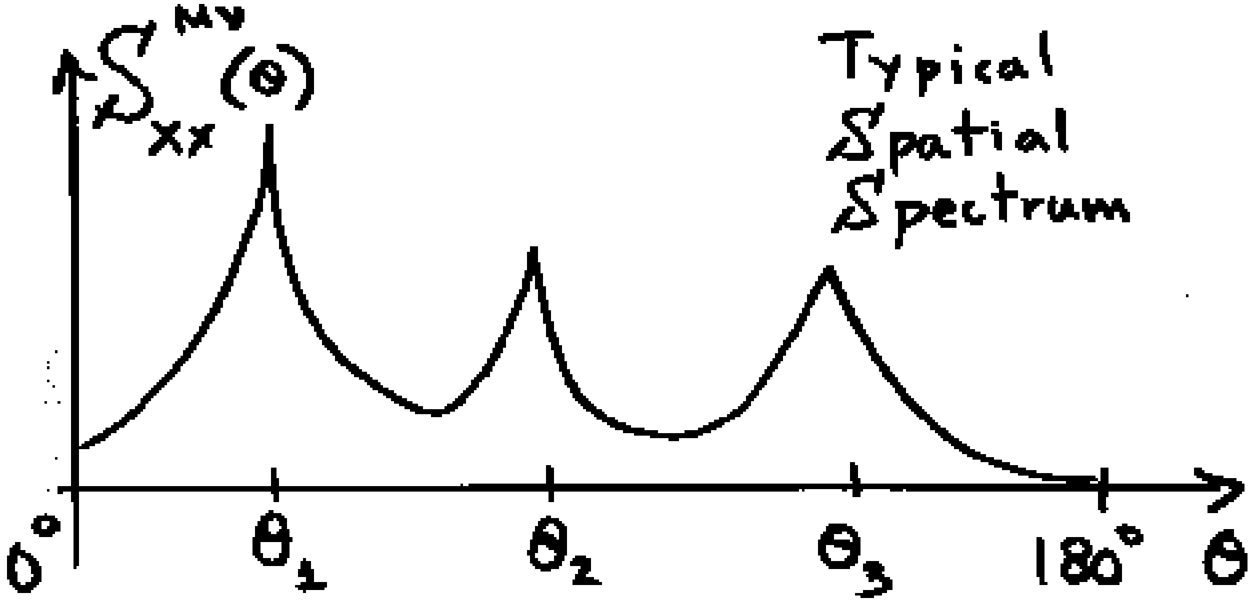
- Recall: Minimum Variance Spatial Spectrum Estimation

$$S_{xx}^{MV}(\mu) = \frac{1}{\underline{s}^H(\mu)\, \hat{\underline{R}}_{xx}^{-1}\, \underline{s}(\mu)}$$

- or directly in terms of angle-of-arrival

$$S_{xx}^{MV}(\theta) = \frac{1}{\underline{a}^H(\theta)\, \hat{\underline{R}}_{xx}^{-1}\, \underline{a}(\theta)}$$

$$\underline{a}(\theta) = \left[1, e^{j\frac{2\pi}{\lambda_c} d\cos\theta}, \ldots, e^{j\frac{(M-1)}{\lambda_c}2\pi d\cos\theta}\right]^T$$



- heights of peak should reflect the relative power levels of the different incident signals

- Except when:
  - two signals are too closely-spaced in angle-of-arrival
  - signal arriving at θ_k is correlated with signal arriving at θ_ℓ, i.e., when $E\{s_k[n]\,s_\ell^*[n]\} \neq 0$ for some $k \neq \ell$

- MUSIC Method :
  - specifically tailored for the case where the incident signals arrive at discrete angles
  - parametric technique premised on the model:

$$\underline{x}[n] = \sum_{k=1}^{P} s_k[n]\, \underline{s}(\mu_k) + \underline{\nu}[n]$$

- examine the structure of $\underline{R}_{xx} = E\{\underline{x}[n]\,\underline{x}^H[n]\}$ under noiseless conditions
  - factor noise in later

- $\underset{M\times 1}{\underline{x}[n]} = s_1[n]\,\underline{s}(\mu_1) + s_2[n]\,\underline{s}(\mu_2) + \dots + s_p[n]\,\underline{s}(\mu_p)$

M = no. of antennas;

- $\forall n$:

$$\underline{x}[n] = \text{span}\{\underline{s}(\mu_1), \underline{s}(\mu_2), \ldots, \underline{s}(\mu_P)\}$$

= signal subspace

= P-dimensional subspace of M-dimensional space (assuming $P<M$)

- let $\{\underline{e}_{P+1}, \underline{e}_{P+2}, \ldots, \underline{e}_M\}$ be a set of $(M-P)$-dim. subspace that is the orthogonal complement of signal subspace

- this implies:

$$\underline{s}^H(\mu_k)\,\underline{e}_i = 0 \quad \text{for } k=1,\dots,P$$
$$\text{and } i=P+1,\dots,M$$

- $\underline{x}[n] = \underline{A}\;\underline{s}[n]$

$$\underset{M\times P}{\underline{A}} = \left[\underline{s}(\mu_1), \underline{s}(\mu_2), \dots, \underline{s}(\mu_P)\right]$$

$$\underset{P\times 1}{\underline{s}[n]} = \left[s_1[n], s_2[n], \dots, s_P[n]\right]^T$$

- Substituting into

$$\underline{\hat{R}}_{xx} = \frac{1}{N}\sum_{n=0}^{N-1} \underline{x}[n]\,\underline{x}^H[n]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} \underline{A}\,\underline{s}[n]\,\underline{s}^H[n]\,A^H$$

$$= \underline{A}\,\underline{\hat{R}}_{ss}\,\underline{A}^H$$

- where $\underline{R}_{ss}$ is the $P \times P$ matrix

$$\underline{\hat{R}}_{ss} = \frac{1}{N}\sum_{n=0}^{N-1} \underline{s}[n]\,\underline{s}^H[n]$$

• consider:

$$\underline{\hat{R}}_{xx}\,\underline{e}_i = \underline{A}\,\underline{\hat{R}}_{ss}\,\underline{A}^H\,\underline{e}_i$$

$$= \underline{A}\,\underline{\hat{R}}_{ss}\begin{bmatrix}\underline{s}^H(\mu_1)\,\underline{e}_i\\ \underline{s}^H(\mu_2)\,\underline{e}_i\\ \vdots\\ \underline{s}^H(\mu_p)\,\underline{e}_i\end{bmatrix}$$

$$= \underline{A}\,\underline{\hat{R}}_{ss}\,\underline{0}$$

$$= \underline{0} = 0\cdot\underline{e}_i \qquad \boxed{\underline{\hat{R}}_{xx}\,\underline{e}_i = 0\cdot\underline{e}_i}$$

- thus, each of the M-P vectors $\underline{e}_i$, $i = P+1, \ldots, M$, which span the orthogonal complement of the signal subspace, is an eigenvector of $\underline{\hat{R}}_{xx}$ associated with the eigenvalue $\lambda = 0$
- Consider $\underline{e}_i$, $i = 1, 2, \ldots, M$ to be the eigenvectors of $\underline{\hat{R}}_{xx}$

- $\hat{\underline{R}}_{xx}\, \underline{e}_i = \lambda_i\, \underline{e}_i \quad , \; i = 1, \ldots, M$

$$\lambda_i = \frac{\underline{e}_i^H\, \hat{\underline{R}}_{xx}\, \underline{e}_i}{\underline{e}_i^H\, \underline{e}_i}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \underline{e}_i^H \underline{x}[n]\, \overbrace{\underline{x}^H[n]\, \underline{e}_i}^{\underline{e}_i^T \underline{x}^*[n]} \Big/ \underline{e}_i^H \underline{e}_i$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left| \underline{e}_i^H \underline{x}[n] \right|^2 \Big/ \underline{e}_i^H \underline{e}_i$$

- thus guaranteed $\lambda_i \geq 0$ for $i = 1, \ldots, M$

- $\lambda = 0$ is the <u>smallest</u> eigenvalue of $\hat{\underline{R}}_{xx}$ of multiplicity of M-P
- the associated eigenvectors satisfy

$$\underline{s}^H(\mu_k)\,\underline{e}_i = 0 \quad \text{for } k = 1, \ldots, P \text{ and } i = P+1, \ldots, M$$

- assuming eigenvalues indexed in descending order

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots > \lambda_{P+1} = \lambda_{P+2} = \ldots = \lambda_M = 0$$

- Factor in the effects of spatially white noise
- $\underline{x}[n] = \underline{A}\ \underline{s}[n] + \underline{\nu}[n]$
- assuming noise to be uncorrelated with the incident signals

$$\underline{R}_{xx} = E\{\underline{x}[n]\,\underline{x}^H[n]\}$$
$$= \underline{A}\,\underline{R}_{ss}\,\underline{A}^H + \underline{R}_{\nu\nu}$$

- where: $\underline{R}_{ss} = E\{\underline{s}[n]\,\underline{s}^H[n]\}$

- assume noise to be uncorrelated from antenna to antenna and of equal power at each antenna.

$$\underline{R}_{\nu\nu} = E\{\underline{\nu}[n]\,\underline{\nu}^H[n]\}$$

$$\underline{R}_{\nu\nu_{mm'}} = E\{\nu[m;n]\,\nu^*[m';n]\}$$

$$= \sigma_w^2\,\delta[m-m'].$$

$$\underline{R}_{\nu\nu} = \sigma_w^2\,\underline{\underline{I}}_M$$

$$\underline{R}_{xx} = \underline{A}\,\underline{R}_{ss}\,\underline{A}^{H} + \sigma_w^2\,\underline{I}_M$$

$$\left|\underline{R}_{xx} - \lambda\,\underline{I}_M\right| = \left|\underline{A}\,\underline{R}_{ss}\,\underline{A}^{H} - \underbrace{(\lambda - \sigma_w^2)}_{\lambda'}\underline{I}_M\right|$$

- λ_i' is an eigenvalue of $\underline{A}\,\underline{R}_{ss}\,\underline{A}^{H}$
- $\lambda_i' = \lambda_i - \sigma_w^2 \implies \lambda_i = \lambda_i' + \sigma_w^2$
- $\lambda_{min} = \sigma_w^2$ is smallest eigenvalue of $\underline{R}_{xx}$ of multiplicity M-P

• no effect on eigenvectors:

$$\underline{R}_{xx}\underline{e}_i = \lambda_i \underline{e}_i$$

$$\left\{\underline{A}\,\underline{R}_{ss}\underline{A}^H + \sigma_w^2 \underline{I}_M\right\}\underline{e}_i = \lambda_i \underline{e}_i$$

$$\left\{\underline{A}\,\underline{R}_{ss}\underline{A}^H\right\}\underline{e}_i = (\lambda_i - \sigma_w^2)\,\underline{e}_i$$

$$i = 1, \dots, M$$