EE648 (CC761-M) DSP II

Session 16 (Live: 3/4/99)

Outline

- Eigenanalysis Algorithms for Spatial Spectrum Estimation
  - Sect. 12.5 of Proakis & Manolakis
  - MUSIC: Sect. 12.5.2 – 12.5.3
  - ESPRIT: Sect. 12.5.4
- Note: Minimum Variance Spectrum Estimation – Sect. 12.4

- continue development of MUSIC Algorithm
- so far: $\underline{R}_{xx} = E\{\underline{x}[n]\,\underline{x}^H[n]\}$
- $\underline{R}_{xx} = \underline{E}\,\underline{\Lambda}\,\underline{E}^{-1} = \underline{E}\,\underline{\Lambda}\,\underline{E}^{H}$

$$= \sum_{i=1}^{M} \lambda_i\, \underline{e}_i\, \underline{e}_i^H \quad \left.\right\} \text{eigenvalue-eigenvector decomposition}$$

- since $\underline{R}_{xx}^H = \underline{R}_{xx} \Rightarrow$ Hermitian-symmetric

- assume eigenvalues ordered as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M$$

- showed last time:

when $\underline{x}[n] = \sum_{k=1}^{P} s_k[n]\, \underline{s}(\mu_k) + \underline{\nu}[n]$

- where: $\underline{R}_{\nu\nu} = E\{\underline{\nu}[n]\underline{\nu}^H[n]\}$

$$= \sigma_w^2 \, \underline{I}_M$$

- note: MUSIC ostensibly not affected by temporal structure of noise

• if $P < M$:

$$\lambda_1 > \lambda_2 > \ldots > \lambda_P > \lambda_{P+1} = \lambda_{P+2} = \ldots = \lambda_M$$
$$= \sigma_w^2$$

• showed:

$$\underline{s}^H(\mu_k)\,\underline{e}_i = 0 \qquad \text{for } k = 1, \ldots, P$$
$$i = P+1, \ldots, M$$

• $\{\underline{e}_{P+1}, \underline{e}_{P+2}, \ldots, \underline{e}_M\}$: "noise" eigenvectors

• span $(M-P)$-dimensional noise subspace

$$\underline{s}^H(\mu_k)\underline{e}_i = \sum_{m=0}^{M-1} e_i[m]\, e^{-j\mu_k m}$$

$$= 0 \quad \text{for } k = 1, \ldots, P$$

$$i = P+1, \ldots, M$$

- each noise eigen~~filter~~ vector can be viewed as an FIR spatial filter whose DSFT is zero at $\mu = \mu_k$, $k = 1, \ldots, P$

- only the nulls at the "true" P spatial frequencies are common to all M−P noise eigenfilter freq. responses
- motivates creation of "artificial" spatial spectrum

$$\hat{S}^{MUSIC}_{xx}(\mu) = \frac{1}{\sum_{i=P+1}^{M} \left| \sum_{m=0}^{M-1} e_i[m] e^{-j\mu m} \right|^2}$$

• alternative construction:

$$\hat{S}_{xx}^{MUSIC}(\mu) = \frac{1}{\sum_{i=P+1}^{M} \left| \underline{s}^H(\mu)\,\underline{e}_i \right|^2}$$

$$= \frac{1}{\sum_{i=P+1}^{M} \underline{s}^H(\mu)\,\underline{e}_i\,\underline{e}_i^H\,\underline{s}(\mu)}$$

$$= \frac{1}{\underline{s}^H(\mu)\left\{\sum_{i=P+1}^{M} \underline{e}_i\,\underline{e}_i^H\right\}\underline{s}(\mu)}$$

$$\underline{P}_N \triangleq \sum_{i=p+1}^{M} \underline{e}_i \underline{e}_i^H$$

$$= \underline{E}_N \underline{E}_N^H$$

· where: $\underset{M \times (M-p)}{\underline{E}_N} = \left[\underline{e}_{p+1} \,\vdots\, \underline{e}_{p+2} \,\vdots\, \cdots \,\vdots\, \underline{e}_M \right]$

· $\underline{P}_N$ is projector operator onto noise subspace $= \text{span}\left\{ \underline{e}_{p+1}, \ldots, \underline{e}_M \right\}$

$= \text{range}\left\{ \underline{E}_N \right\}$

• note: $\underline{P}_N^2 = \underline{P}_N \underline{P}_N = \underline{P}_N$

$\underline{P}_N^H = \underline{P}_N$

• for any $M \times 1$ vector $\underline{y}$

$\underline{P}_N \underline{y}$ = projection of $\underline{y}$ onto noise subspace

$$\hat{S}_{xx}^{MUSIC}(u) = \frac{1}{\underline{s}^H(u)\, \underline{P}_N\, \underline{s}(u)}$$

- in terms of angle-of-arrival:

$$\hat{S}_{xx}^{MUSIC}(\theta) = \frac{1}{\underline{a}^H(\theta)\, \underline{P}_N\, \underline{a}(\theta)}$$

- where:

$$\underline{a}(\theta) = \left[1, e^{j\frac{2\pi}{\lambda} d\cos\theta}, \ldots, e^{j\frac{(M-1)}{\lambda} 2\pi d\cos\theta}\right]^T$$

$\left(\text{recall: } \mu = \frac{2\pi}{\lambda} d\cos\theta\right)$

- See MUSICDemo.m at web site

- Advantages of MUSIC over Minimum Variance
- better/higher resolution: sharper spectral peaks
- not as affected by correlation between signals arriving from different directions
- alternatively compute arrival angles via the roots of a polynomial
    - as opposed to spectral search

- Estimation of SIgnal Parameters by Rotational Invariance Techniques (ESPRIT)
- signal subspace

$= \text{span}\{\underline{e}_1, \underline{e}_2, \dots, \underline{e}_P\}$

$= \text{span}\{\underline{s}(\mu_1), \underline{s}(\mu_2), \dots, \underline{s}(\mu_P)\}$

$\underline{E}_s = [\underline{e}_1, \underline{e}_2, \dots, \underline{e}_P]$

recall: $\underline{S} = [\underline{s}(\mu_1), \dots, \underline{s}(\mu_P)]$

• since range$\{\underline{E}_s\}$ = range$\{\underline{S}'\}$

$$\underset{M\times P}{\underline{S}'} = \underset{M\times P}{\underline{E}_s}\ \underset{P\times P}{\underline{T}} \quad \Big\}\ \text{think column-wise equality}$$

$$\underline{s}(\mu_k) = \underline{E}_s\,\underline{t}_k = \sum_{i=1}^{P} \underline{e}_i\, t_{ik}$$

$$\underline{T} = [\underbrace{\underline{t}_1}_{P\times 1}, \underline{t}_2, \ldots, \underline{t}_P] \qquad k = 1, \ldots, P$$

- $\underline{s}(\mu_k) = \left[1, e^{j\mu_k}, e^{j2\mu_k}, \ldots, e^{j(M-1)\mu_k}\right]^T$

- define: first $(M-1)\times 1$ sub-vector:

$$\underline{s}^{(1)}(\mu_k) = \left[1, e^{j\mu_k}, \ldots, e^{j(M-2)\mu_k}\right]^T$$

- second $(M-1)\times 1$ sub-vector:

$$\underline{s}^{(2)}(\mu_k) = \left[e^{j\mu_k}, \ldots, e^{j(M-1)\mu_k}\right]^T$$

- observe: $\underline{s}^{(2)}(\mu_k) = e^{j\mu_k}\,\underline{s}^{(1)}(\mu_k)$

• define $(M-1)\times M$ selection matrices:

$$\underline{\Gamma}_1 = \left[\underline{I}_{M-1} \quad \underline{0}_{(M-1)\times 1}\right]; \quad \underline{\Gamma}_2 = \left[\underline{0}_{(M-1)\times 1} \quad \underline{I}_{M-1}\right]$$

$$\underline{s}^{(1)}(\mu) = \underline{\Gamma}_1\, \underline{s}(\mu); \quad \underline{s}^{(2)}(\mu) = \underline{\Gamma}_2\, \underline{s}(\mu)$$

$$\left\{\underline{\Gamma}_1\, \underline{s}(\mu_k)\right\} e^{j\mu_k} = \underline{\Gamma}_2\, \underline{s}(\mu_k)$$

$$k = 1, \ldots, P$$

$$\left\{\underline{\Gamma}_1\, \underline{S}\right\} \underline{\underline{\Phi}} = \underline{\Gamma}_2\, \underline{S}$$