EE648 (CC761-M) DSP II

Session 16 (Live: 3/4/99)

Outline

- Eigenanalysis Algorithms for Spatial Spectrum Estimation
  - Sect. 12.5 of Proakis & Manolakis
  - MUSIC: Sect. 12.5.2 – 12.5.3
  - ESPRIT: Sect. 12.5.4
- Note: Minimum Variance Spectrum Estimation – Sect. 12.4

- continue development of MUSIC Algorithm
- so far: $\underline{R}_{xx} = E\{\underline{x}[n]\,\underline{x}^H[n]\}$
- $\underline{R}_{xx} = \underline{E}\,\underline{\Lambda}\,\underline{E}^{-1} = \underline{E}\,\underline{\Lambda}\,\underline{E}^H$

$$= \sum_{i=1}^{M} \lambda_i\, \underline{e}_i\, \underline{e}_i^H \quad \Big\}\ \text{eigenvalue-eigenvector decomposition}$$

- since $\underline{R}_{xx}^H = \underline{R}_{xx} \Rightarrow$ Hermitian-symmetric

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- assume eigenvalues ordered as

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_M$$

- showed last time:

  when $\underline{x}[n] = \sum_{k=1}^{P} s_k[n]\,\underline{s}(\mu_k) + \underline{\nu}[n]$

- where: $\underline{R}_{\nu\nu} = E\{\underline{\nu}[n]\,\underline{\nu}^H[n]\}$

$$= \sigma_w^2\, \underline{I}_M$$

- note: MUSIC ostensibly not affected by temporal structure of noise

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- if $P < M$:

$$\lambda_1 > \lambda_2 > \dots > \lambda_P > \lambda_{P+1} = \lambda_{P+2} = \dots = \lambda_M = \sigma_w^2$$

- showed:

$$\underline{s}^H(\mu_k)\,\underline{e}_i = 0 \quad \text{for } k = 1, \dots, P;\ i = P+1, \dots, M$$

- $\{\underline{e}_{P+1}, \underline{e}_{P+2}, \dots, \underline{e}_M\}$: "noise" eigenvectors
- span $(M-P)$-dimensional noise subspace

$$\underline{s}^H(\mu_k)\underline{e}_i = \sum_{m=0}^{M-1} e_i[m]\, e^{-j\mu_k m}$$

$$= 0 \quad \text{for } k = 1, \ldots, P$$

$$i = P+1, \ldots, M$$

- each noise eigen~~filter~~ vector can be viewed as an FIR spatial filter whose DSFT is zero at $\mu = \mu_k$, $k = 1, \ldots, P$

- only the nulls at the "true" P spatial frequencies are common to all M−P noise eigenfilter freq. responses
- motivates creation of "artificial" spatial spectrum

$$\hat{S}_{xx}^{MUSIC}(\mu) = \frac{1}{\sum_{i=P+1}^{M} \left| \sum_{m=0}^{M-1} e_i[m] e^{-j\mu m} \right|^2}$$

- alternative construction:

$$\hat{S}_{xx}^{MUSIC}(\mu) = \frac{1}{\sum_{i=P+1}^{M} \left| \underline{s}^H(\mu)\underline{e}_i \right|^2}$$

$$= \frac{1}{\sum_{i=P+1}^{M} \underline{s}^H(\mu)\underline{e}_i \; \underline{e}_i^H \underline{s}(\mu)}$$

$$= \frac{1}{\underline{s}^H(\mu)\left\{\sum_{i=P+1}^{M} \underline{e}_i \underline{e}_i^H\right\}\underline{s}(\mu)}$$

$$\underline{P}_N \triangleq \sum_{i=P+1}^{M} \underline{e}_i \underline{e}_i^H$$

$$= \underline{E}_N \underline{E}_N^H$$

- where: $\underset{M \times (M-P)}{\underline{E}_N} = [\underline{e}_{P+1} \,\vdots\, \underline{e}_{P+2} \,\vdots\, \cdots \,\vdots\, \underline{e}_M]$

- $\underline{P}_N$ is projector operator onto noise subspace $= \text{span}\{\underline{e}_{P+1}, \ldots, \underline{e}_M\}$

$$= \text{range}\{\underline{E}_N\}$$

- note: $\underline{P}_N^2 = \underline{P}_N \underline{P}_N = \underline{P}_N$

  $\underline{P}_N^H = \underline{P}_N$

- for any $M\times 1$ vector $\underline{y}$

  $\underline{P}_N \underline{y}$ = projection of $\underline{y}$ onto noise subspace

$$\hat{S}_{xx}^{MUSIC}(\mu) = \frac{1}{\underline{s}^H(\mu)\,\underline{P}_N\,\underline{s}(\mu)}$$

- in terms of angle-of-arrival:

$$\hat{S}_{xx}^{MUSIC}(\theta) = \frac{1}{\underline{a}^H(\theta)\,\underline{P}_N\,\underline{a}(\theta)}$$

- where:

$$\underline{a}(\theta) = \left[1, e^{j\frac{2\pi}{\lambda}d\cos\theta}, \ldots, e^{j\frac{(M-1)2\pi d\cos\theta}{\lambda}}\right]^T$$

(recall: $\mu = \frac{2\pi}{\lambda} d\cos\theta$)

- See MUSICDemo.m at web site

- Advantages of MUSIC over Minimum Variance
- better/higher resolution: sharper spectral peaks
- not as affected by correlation between signals arriving from different directions
- alternatively compute arrival angles via the roots of a polynomial
  - as opposed to spectral search

- Estimation of SIgnal Parameters by Rotational Invariance Techniques (ESPRIT)
- signal subspace

  $= \text{span}\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_P\}$

  $= \text{span}\{\underline{s}(\mu_1), \underline{s}(\mu_2), \ldots, \underline{s}(\mu_P)\}$

$\underline{E}_s = [\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_P]$

recall: $\underline{S} = [\underline{s}(\mu_1), \ldots, \underline{s}(\mu_P)]$

- since $\text{range}\{\underline{E}_s\} = \text{range}\{\underline{S}\}$

$$\underset{M\times P}{\underline{S}} = \underset{M\times P}{\underline{E}_s}\ \underset{P\times P}{\underline{T}} \quad \left.\right\} \text{think column-wise equality}$$

$$\underline{s}(\mu_k) = \underline{E}_s\,\underline{t}_k = \sum_{i=1}^{P} \underline{e}_i\, t_{ik}$$

$$\underline{T} = [\underset{P\times 1}{\underline{t}_1}, \underline{t}_2, \dots, \underline{t}_P] \qquad k = 1, \dots, P$$

- $\underline{s}(\mu_k) = [1, e^{j\mu_k}, e^{j2\mu_k}, \dots, e^{j(M-1)\mu_k}]^T$
- define: first $(M-1)\times 1$ sub-vector:

$$\underline{s}^{(1)}(\mu_k) = [1, e^{j\mu_k}, \dots, e^{j(M-2)\mu_k}]^T$$

- second $(M-1)\times 1$ sub-vector:

$$\underline{s}^{(2)}(\mu_k) = [e^{j\mu_k}, \dots, e^{j(M-1)\mu_k}]^T$$

- observe: $\underline{s}^{(2)}(\mu_k) = e^{j\mu_k}\,\underline{s}^{(1)}(\mu_k)$

- define $(M-1)\times M$ selection matrices:

$$\underline{\Gamma}_1 = [\underline{I}_{M-1}\ \ \underline{0}_{(M-1)\times 1}]; \quad \underline{\Gamma}_2 = [\underline{0}_{(M-1)\times 1}\ \ \underline{I}_{M-1}]$$

$$\underline{s}^{(1)}(\mu) = \underline{\Gamma}_1\,\underline{s}(\mu); \quad \underline{s}^{(2)}(\mu) = \underline{\Gamma}_2\,\underline{s}(\mu)$$

$$\{\underline{\Gamma}_1\,\underline{s}(\mu_k)\}\, e^{j\mu_k} = \underline{\Gamma}_2\,\underline{s}(\mu_k) \qquad k = 1, \dots, P$$

$$\{\underline{\Gamma}_1\,\underline{S}\}\,\underline{\Phi} = \underline{\Gamma}_2\,\underline{S}$$