EE648 (CC761-M) DSP II
Session 17 (Live: 3/9/99)

Outline

- ESPRIT Method for Angle-of-Arrival Estimation
  - Sect. 12.5.4 of P&M
    ESPRIT for dual problem of estimating frequencies of sinewaves in noise

- $\underline{E}_s = [\underline{e}_1 \vdots \underline{e}_2 \vdots \cdots \vdots \underline{e}_p]$
  - columns are "signal" eigenvectors of spatial correlation matrix $\underline{R}_{xx}$
- ideall: $\underset{\sim}{\underline{S}} = \underline{E}_s \underset{p \times p}{\underline{T}} \quad \left(\underline{E}_s = \underline{S}\,\underline{T}^{-1}\right)$

where: $\underset{\sim}{\underline{S}} = [\underline{s}(\mu_1) \vdots \underline{s}(\mu_2) \vdots \cdots \vdots \underline{s}(\mu_p)]$

$$\underline{s}(\mu) = \left[1, e^{j\mu}, e^{j2\mu}, \ldots, e^{j(M-1)\mu}\right]^T$$

$$\underline{\Gamma}_1 = \left[\underline{I}_{M-1} \quad \underset{(M-1)\times 1}{\underline{0}} \right] ; \quad \underline{\Gamma}_2 = \left[\underset{(M-1)\times 1}{\underline{0}} \quad \underline{I}_{M-1} \right]$$

• VIP property :

$$\underline{\Gamma}_1 \underline{s}(\mu_k) e^{j\mu_k} = \underline{\Gamma}_2 \underline{s}(\mu_k)$$

$$k = 1, \ldots, P$$

$$\underline{\Gamma}_1 \underline{S} \underline{\Phi} = \underline{\Gamma}_2 \underline{S} \quad \left(\begin{array}{c}\text{column-}\\ \text{wise}\\ \text{equality}\end{array}\right)$$

$$\underset{P\times P}{\underline{\Phi}} = \text{diag}\left\{ e^{j\mu_1}, e^{j\mu_2}, \ldots, e^{j\mu_P} \right\}$$

• Substituting $\underline{E}_s = \underline{S}\,\underline{T}^{-1}$ $(\underline{S} = \underline{E}_s \underline{T})$

$$\underline{\Gamma}_1 (\underline{E}_s \underline{T})\, \underline{\underline{\Phi}} = \underline{\Gamma}_2 (\underline{E}_s \underline{T})$$

• Post-multiply by $\underline{T}^{-1}$ on both sides

$$\underline{\Gamma}_1 \underline{E}_s \left\{ \underline{T}\, \underline{\underline{\Phi}}\, \underline{T}^{-1} \right\} = \underline{\Gamma}_2 \underline{E}_s$$

• define: $\underline{\Psi} = \underline{T}\, \underline{\underline{\Phi}}\, \underline{T}^{-1}$ } $\underline{\underline{\Phi}}$ is diagonal matrix

$\underline{\Psi} = \underline{T}\,\underline{\Phi}\,\underline{T}^{-1}$ is in form of eigenvalue decomposition

- thus: $e^{j\mu_k}$, $k = 1, \ldots, P$ are the eigenvalues of $\underline{\Psi}$ $\Rightarrow$ which is the solution to

$$\underline{A}\,\underline{X} = \underline{B}$$

$$\underline{A} = \underline{\Gamma}_1 \underline{E}_s \;;\; \underline{B} = \underline{\Gamma}_2 \underline{E}_s \;;\; \underline{X} = \underline{\Psi}$$

- $\underline{A}\,\underline{X} = \underline{B}$

$(M-1)\times P \quad P\times P \quad (M-1)\times P$

- assuming $M > P$ => this represents overdetermined systems of equations
- least square error soln: in matlab:

$$\underline{X} = (\underline{A}^H \underline{A})^{-1} \underline{A}^H \underline{B} \quad (X = A \backslash B)$$

- $\underline{\Psi} = (\underline{E}_s^H \underline{\Gamma}_1^T \underline{\Gamma}_1 \underline{E}_s)^{-1} \underline{E}_s \underline{\Gamma}_1 \underline{\Gamma}_2 \underline{E}_s$

- Summary of ESPRIT
- 1. Compute P signal eigenvectors of $\underline{R}_{xx}$ (assoc. with P largest eigenvalues)

$$\underline{E}_s = [\underline{e}_1 \mid \underline{e}_2 \mid \cdots \mid \underline{e}_P]$$

2. Solve $(\underline{T}_1 \underline{E}_s)\underline{\Psi} = (\underline{T}_2 \underline{E}_s)$
3. Compute P eigenvalues of $\underline{\Psi}$
   - ideally: eigenvalues are $e^{j\mu_k}$ $k = 1, \dots, P$

- ESPRIT Example

- $P = 2$ planewaves incident upon $M = 3$ antennas

- Given 3×3 spatial correlation matrix:

$$\underline{R}_{xx} = \begin{bmatrix} 1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 \end{bmatrix}$$

- (asymptotic form)

$$\left|\underline{R}_{xx}-\lambda\underline{I}_3\right| = \begin{vmatrix} 1-\lambda & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1-\lambda & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left\{(1-\lambda)^2-\frac{1}{2}\right\}-\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(1-\lambda)$$

$$= (1-\lambda)\left\{1-2\lambda+\lambda^2-\frac{1}{2}-\frac{1}{2}\right\}$$

$$\left.\begin{aligned} &= (1-\lambda)\left\{-2\lambda+\lambda^2\right\} \\ &= \lambda(1-\lambda)(\lambda-2)=0 \end{aligned}\right\} \begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 1 \\ \lambda_3 &= 0 \end{aligned}$$

$$\lambda_1 = 2: \quad \left\{ \underline{R}_{xx} - 2\,\underline{I}_3 \right\} \underline{e}_1 = \underline{0}_{3\times 1}$$

$$\begin{bmatrix} -1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{21} \\ e_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

set $e_{11} = 1$: $$\begin{bmatrix} -1 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} e_{21} \\ e_{31} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$e_{31} = \frac{e_{21}}{\sqrt{2}} \quad \Rightarrow \quad -e_{21} + \frac{1}{2} e_{21} = -\frac{1}{\sqrt{2}}$$

$$e_{21} = \sqrt{2} \qquad \underline{e}_1 = \begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix}^T$$

$$\lambda_2 = 1: \quad \left\{ \underline{R}_{xx} - \underline{I} \right\} \underline{e}_2 = 0$$

$$\begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} e_{12} \\ e_{22} \\ e_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- set $e_{12} = 1$ $\quad \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} e_{22} \\ e_{32} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \end{bmatrix}$

- $e_{22} = 0$ $\quad \frac{1}{\sqrt{2}} e_{32} = -\frac{1}{\sqrt{2}} \Rightarrow e_{32} = -1$

$$\underline{e}_2 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$$

$$\underline{E}_s = \left[\underline{e}_1 \vdots \underline{e}_2\right] = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & 0 \\ 1 & -1 \end{bmatrix}$$

$$(\underline{I}_1 \underline{E}_s)\Psi = (\underline{I}_2 \underline{E}_s)$$

$$\begin{bmatrix} 1 & 1 \\ \sqrt{2} & 0 \end{bmatrix} \underline{\Psi} = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\underline{\Psi} = -\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -1 \\ -\sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 1 & -1 \end{bmatrix}$$

$$\underline{\Psi} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$|\Psi - \lambda \underline{I}_2| = \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & -1/\sqrt{2} \\ 1/\sqrt{2} & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2}} - \lambda\right)^2 + \frac{1}{2} = \lambda^2 - \sqrt{2}\lambda + \frac{1}{2} + \frac{1}{2}$$

$$= \lambda^2 - \sqrt{2}\lambda + 1$$

$$\lambda = \frac{+\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}$$

- eigenvalues: $e^{+j\frac{\pi}{4}}, e^{-j\frac{\pi}{4}}$

- assuming $d = \lambda_c/2$, then

$$u = \pi\cos\theta \Rightarrow \theta = \cos^{-1}(u/\pi)$$

$$e^{j\frac{\pi}{4}} = e^{j\pi\cos\theta_1} = e^{ju_1}$$

$$\theta_1 = \cos^{-1}(1/4) = 75.52^\circ$$

$$e^{-j\frac{\pi}{4}} = e^{j\pi\cos\theta_2} = e^{ju_2}$$

$$\theta_2 = \cos^{-1}(-1/4) = 104.48^\circ$$