# EE648 (CC761-M) DSP II
## Session 17 (Live: 3/9/99)

**Outline**

- ESPRIT Method for Angle-of-Arrival Estimation
  - Sect. 12.5.4 of P&M ESPRIT for dual problem of estimating frequencies of sinewaves in noise

- $\underline{\underline{E}}_s = [\underline{e}_1 \vdots \underline{e}_2 \vdots \cdots \vdots \underline{e}_P]$
- columns are "signal" eigenvectors of spatial correlation matrix $\underline{\underline{R}}_{xx}$
- ideall: $\underline{\underline{S}} = \underline{\underline{E}}_s \underset{P\times P}{\underline{\underline{T}}}$ $\quad (\underline{\underline{E}}_s = \underline{\underline{S}}\,\underline{\underline{T}}^{-1})$

where: $\underline{\underline{S}} = [\underline{s}(\mu_1) \vdots \underline{s}(\mu_2) \vdots \cdots \vdots \underline{s}(\mu_P)]$

$\underline{s}(\mu) = [1, e^{j\mu}, e^{j2\mu}, \ldots, e^{j(M-1)\mu}]^T$

$$\underline{\underline{\Gamma}}_1 = [\underline{\underline{I}}_{M-1} \;\; \underline{0}_{(M-1)\times 1}] \; ; \; \underline{\underline{\Gamma}}_2 = [\underline{0}_{(M-1)\times 1} \;\; \underline{\underline{I}}_{M-1}]$$

- VIP property:

$$\underline{\underline{\Gamma}}_1 \underline{s}(\mu_k) e^{j\mu_k} = \underline{\underline{\Gamma}}_2 \underline{s}(\mu_k) \qquad k = 1, \ldots, P$$

$$\underline{\underline{\Gamma}}_1 \underline{\underline{S}}\,\underline{\underline{\Phi}} = \underline{\underline{\Gamma}}_2 \underline{\underline{S}} \quad \text{(column-wise equality)}$$

$$\underset{P\times P}{\underline{\underline{\Phi}}} = \text{diag}\{e^{j\mu_1}, e^{j\mu_2}, \ldots, e^{j\mu_P}\}$$

- Substituting $(\underline{\underline{S}} = \underline{\underline{E}}_s \underline{\underline{T}})$ $\quad \underline{\underline{E}}_s = \underline{\underline{S}}\,\underline{\underline{T}}^{-1}$

$$\underline{\underline{\Gamma}}_1 (\underline{\underline{E}}_s \underline{\underline{T}})\, \underline{\underline{\Phi}} = \underline{\underline{\Gamma}}_2 (\underline{\underline{E}}_s \underline{\underline{T}})$$

- Post-multiply by $\underline{\underline{T}}^{-1}$ on both sides

$$\underline{\underline{\Gamma}}_1 \underline{\underline{E}}_s \{\underline{\underline{T}}\,\underline{\underline{\Phi}}\,\underline{\underline{T}}^{-1}\} = \underline{\underline{\Gamma}}_2 \underline{\underline{E}}_s$$

- define: $\underline{\underline{\Psi}} = \underline{\underline{T}}\,\underline{\underline{\Phi}}\,\underline{\underline{T}}^{-1}$ $\}$ $\underline{\underline{\Phi}}$ is diagonal matrix

$\underline{\Psi} = \underline{T}\,\underline{\Phi}\,\underline{T}^{-1}$ is in form of eigenvalue decomposition

- thus: $e^{j\mu_k}$, $k = 1, \ldots, P$ are the eigenvalues of $\underline{\Psi}$ => which is the solution to

$$\underline{A}\,\underline{X} = \underline{B}$$

$$\underline{A} = \underline{\Gamma}_1 \underline{E}_s \; ; \; \underline{B} = \underline{\Gamma}_2 \underline{E}_s \; ; \; \underline{X} = \underline{\Psi}$$

- $\underset{(M-1)\times P}{\underline{A}} \; \underset{P\times P}{\underline{X}} = \underset{(M-1)\times P}{\underline{B}}$
- assuming $M > P$ => this represents overdetermined systems of equations
- least square error sol'n: in matlab:

$$\underline{X} = (\underline{A}^H \underline{A})^{-1} \underline{A}^H \; \underline{B} \qquad (X = A\backslash B)$$

- $\underline{\Psi} = (\underline{E}_s^H \underline{\Gamma}_1^T \underline{\Gamma}_1 \underline{E}_s)^{-1} \underline{E}_s \underline{\Gamma}_1 \underline{\Gamma}_2 \underline{E}_s$

- Summary of ESPRIT
- 1. Compute $P$ signal eigenvectors of $\underline{R}_{xx}$ (assoc. with $P$ largest eigenvalues)

$$\underline{E}_s = [\underline{e}_1 \,|\, \underline{e}_2 \,|\, \cdots \,|\, \underline{e}_P]$$

2. Solve $(\underline{\Gamma}_1 \underline{E}_s)\underline{\Psi} = (\underline{\Gamma}_2 \underline{E}_s)$
3. Compute $P$ eigenvalues of $\underline{\Psi}$
   - ideally: eigenvalues are $e^{j\mu_k}$, $k = 1, \ldots, P$

- <u>ESPRIT Example</u>
- $P = 2$ planewaves incident upon $M = 3$ antennas
- Given 3x3 spatial correlation matrix:

$$\underline{R}_{xx} = \begin{bmatrix} 1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 \end{bmatrix}$$

- (asymptotic form)

$$\left|\underline{R}_{xx}-\lambda\underline{I}_3\right| = \begin{vmatrix} 1-\lambda & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1-\lambda & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left\{(1-\lambda)^2-\frac{1}{2}\right\} - \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(1-\lambda)$$

$$= (1-\lambda)\left\{1-2\lambda+\lambda^2-\frac{1}{2}-\frac{1}{2}\right\}$$

$$\left.\begin{aligned} &= (1-\lambda)\left\{-2\lambda+\lambda^2\right\} \\ &= \lambda(1-\lambda)(\lambda-2) = 0 \end{aligned}\right\} \begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 1 \\ \lambda_3 &= 0 \end{aligned}$$

$\lambda_1 = 2$: $\left\{\underline{R}_{xx} - 2\underline{I}_3\right\}\underline{e}_1 = \underline{0}_{3\times 1}$

$$\begin{bmatrix} -1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{21} \\ e_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

set $e_{11} = 1$ : $\begin{bmatrix} -1 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} e_{21} \\ e_{31} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \end{bmatrix}$

$e_{31} = \dfrac{e_{21}}{\sqrt{2}} \quad \Rightarrow -e_{21} + \dfrac{1}{2}e_{21} = -\dfrac{1}{\sqrt{2}}$

$e_{21} = \sqrt{2} \quad \underline{e}_1 = \begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix}^T$

$\lambda_2 = 1$: $\left\{\underline{R}_{xx} - \underline{I}\right\}\underline{e}_2 = 0$

$$\begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} e_{12} \\ e_{22} \\ e_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- set $e_{12} = 1$ $\quad \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} e_{22} \\ e_{32} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \end{bmatrix}$
- $e_{22} = 0 \quad \dfrac{1}{\sqrt{2}}e_{32} = -\dfrac{1}{\sqrt{2}} \Rightarrow e_{32} = -1$

$\underline{e}_2 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$

$$\underline{E}_s = \left[\underline{e}_1 \,\vdots\, \underline{e}_2\right] = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & 0 \\ 1 & -1 \end{bmatrix}$$

$$(\underline{\Gamma}_1 \underline{E}_s)\,\underline{\Psi} = (\underline{\Gamma}_2 \underline{E}_s)$$

$$\begin{bmatrix} 1 & 1 \\ \sqrt{2} & 0 \end{bmatrix} \underline{\Psi} = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\underline{\Psi} = -\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -1 \\ -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 1 & -1 \end{bmatrix}$$

$$\underline{\underline{\Psi}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\left| \Psi - \lambda \underline{I}_2 \right| = \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & -1/\sqrt{2} \\ 1/\sqrt{2} & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2}} - \lambda\right)^2 + \frac{1}{2} = \lambda^2 - \sqrt{2}\lambda + \frac{1}{2} + \frac{1}{2}$$

$$= \lambda^2 - \sqrt{2}\lambda + 1$$

$$\lambda = \frac{+\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}$$

- eigenvalues: $e^{+j\frac{\pi}{4}}, e^{-j\frac{\pi}{4}}$
- assuming $d = \lambda_c/2$, then

$$u = \pi \cos\theta \Rightarrow \theta = \cos^{-1}(u/\pi)$$

$$e^{j\frac{\pi}{4}} = e^{j\pi\cos\theta_1} = e^{ju_1}$$

$$\theta_1 = \cos^{-1}(1/4) = 75.52^\circ$$

$$e^{-j\frac{\pi}{4}} = e^{j\pi\cos\theta_2} = e^{ju_2}$$

$$\theta_2 = \cos^{-1}(-1/4) = 104.48^\circ$$