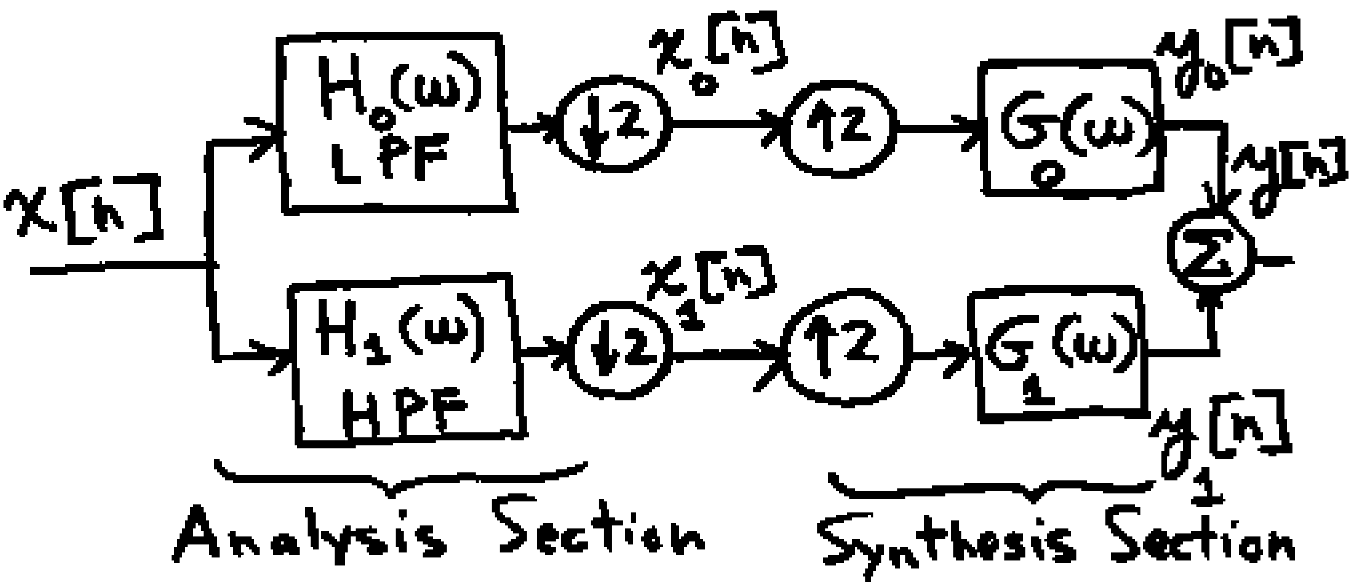
EE648 (CC761-M) DSP II

Session 20 (Live: 3/25/99)

Outline

- Perfect Reconstruction (PR) Filter Banks
  - Two-Channel Case
    - P+M Sect. 10.9.9
    - V - Chap. 5

• Analysis of Two-Channel PR Filter Bank



Desire: $y[n] = \alpha\, x[n-D]$

$$X_0(\omega) = \frac{1}{2} H_0\left(\frac{\omega}{2}\right) X\left(\frac{\omega}{2}\right) + \frac{1}{2} H_0\left(\frac{\omega-2\pi}{2}\right) X\left(\frac{\omega-2\pi}{2}\right)$$

$$X_1(\omega) = \frac{1}{2} H_1\left(\frac{\omega}{2}\right) X\left(\frac{\omega}{2}\right) + \frac{1}{2} H_1\left(\frac{\omega-2\pi}{2}\right) X\left(\frac{\omega-2\pi}{2}\right)$$

$$Y_0(\omega) = G_0(\omega)\, X_0(2\omega) = \frac{1}{2} G_0(\omega) \left\{ H_0(\omega) X(\omega) + H_0(\omega-\pi) X(\omega-\pi) \right\}$$

$$Y_1(\omega) = \frac{1}{2} G_1(\omega) \left\{ H_1(\omega) X(\omega) + H_1(\omega - \pi) X(\omega - \pi) \right\}$$

$$Y(\omega) = Y_0(\omega) + Y_1(\omega)$$

$$= \frac{1}{2} \left\{ H_0(\omega) G_0(\omega) + H_1(\omega) G_1(\omega) \right\} X(\omega)$$

$$+ \underbrace{\frac{1}{2} \left\{ H_0(\omega - \pi) G_0(\omega) + H_1(\omega - \pi) G_1(\omega) \right\}}_{\text{alinsing term}} X(\omega - \pi)$$

• to insure aliasing is zero, common practice (not only possibility, though) is to set:

$$G_0(\omega) = H_1(\omega - \pi) \Rightarrow g_0[n] = (-1)^n h_1[n]$$

$$G_1(\omega) = -H_0(\omega - \pi) \Rightarrow g_1[n] = -(-1)^n h_0[n]$$

In this case:

$$H_0(\omega-\pi)\, G_0(\omega) + H_1(\omega-\pi)\, G_1(\omega) = 0$$

for all ω

(<u>Note</u>: $e^{j\pi n} = (-1)^n$)

• at this point:

$$Y(\omega) = \frac{1}{2}\left\{ \underbrace{H_0(\omega) H_1(\omega - \pi) - H_1(\omega) H_0(\omega - \pi)}_{T(\omega)\ \Rightarrow\ \text{desire} = \alpha e^{-jD\omega}} \right\} X(\omega)$$

• let $h_0[n] \overset{DTFT}{\longleftrightarrow} H_0(\omega)$ be a LPF of length N with symmetric impulse ($h_0[N-1-n] = h_0[n]$, $n = 0, 1, \ldots, N-1$) response guaranteeing linear phase

- in the case of a Quadrature Mirror Filter (QMF) Bank:

$$h_1[n] = (-1)^n h_0[n]$$

$$\Rightarrow H_1(\omega) = H_0(\omega - \pi)$$

$\Rightarrow$ translates $h_0[n]$ into a HPF

- let $h[n]$ is a LPF - symmetric FIR with a cut-off near $\omega_c = \pi/2$

$$h_0[n] = h[n]$$

$$h_1[n] = (-1)^n h[n]$$

- Substituting $H_1(\omega) = H(\omega-\pi)$
  $H_0(\omega) = H(\omega)$

$$T(\omega) = H(\omega)\,H(\omega-2\pi) - H(\omega-\pi)\,H(\omega-\pi)$$
$$= H^2(\omega) - H^2(\omega-\pi)$$

- since $h[n]$ is symmetric:

$$H(\omega) = H_r(\omega)\, e^{-j\left(\frac{N-1}{2}\right)\omega}$$

$$H^2(\omega) = H_r^2(\omega)\, e^{-j(N-1)\omega}$$

$$H^2(\omega-\pi) = H_r^2(\omega-\pi)\, e^{-j(N-1)(\omega-\pi)}$$

$$= (-1)^{N-1} H_r^2(\omega-\pi)\, e^{-j(N-1)\omega}$$

• Substituting:

$$T(\omega) = \frac{1}{2}\left\{ |H(\omega)|^2 - (-1)^{N-1} |H(\omega-\pi)|^2 \right\} \cdot e^{-j(N-1)\omega}$$

$$= A(\omega)\, e^{-j(N-1)\omega}$$

•

- note: if N is odd, then $\omega = \pi/2$:

$$A(\omega)\Big|_{\omega=\frac{\pi}{2}} = \left|H\left(\tfrac{\pi}{2}\right)\right|^2 - \left|H\left(\tfrac{\pi}{2}\right)\right|^2 = 0$$

- Thus, choose N even! In this case:

$$A(\omega) = |H(\omega)|^2 + |H(\omega-\pi)|^2$$

- Ideally: $A(\omega) = 1$ for all ω

- For PR under these QMF conditions, we require a LPF $h[n]$ - symmetric FIR - with cut-off near $\omega_c = \pi/2$ satisfying the power-symmetry constraint

$$|H(\omega)|^2 + |H(\omega-\pi)|^2 = 1$$

What kind of filters satisfy this condition?

• Simplest Example:

$$h[n] = \{\underset{\uparrow}{\tfrac{1}{2}}, \tfrac{1}{2}\} = \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-1]$$

$$H(\omega) = \frac{1}{2} + \frac{1}{2}e^{-j\omega}$$

$$= e^{-j\frac{\omega}{2}}\left\{\frac{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}{2}\right\}$$

$$= \cos\left(\frac{\omega}{2}\right)e^{-j\frac{\omega}{2}}$$

$$|H(\omega)|^2 = \cos^2\left(\frac{\omega}{2}\right)$$

$$H(\omega-\pi) = \cos\left(\frac{\omega-\pi}{2}\right)e^{-j\frac{(\omega-\pi)}{2}}$$

$$= \cos\left(\frac{\omega}{2}-\frac{\pi}{2}\right) j\, e^{-j\frac{\omega}{2}}$$

$$= \sin\left(\frac{\omega}{2}\right) j\, e^{-j\frac{\omega}{2}}$$

$$|H(\omega)|^2 + |H(\omega-\pi)|^2$$

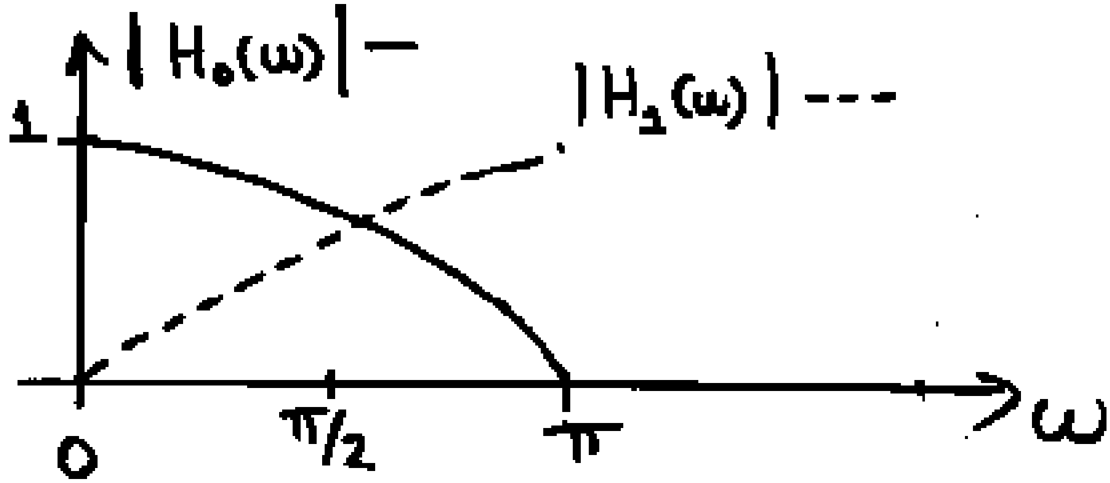
$$= \cos^2\left(\frac{\omega}{2}\right) + \sin^2\left(\frac{\omega}{2}\right) = 1 \quad \forall \omega$$

- satisfies power symmetry constraint!

$$h_0[n] = h[n] \quad ; \quad h_1[n] = (-1)^n h[n]$$

$$H_0(\omega) = H(\omega) \quad ; \quad H_1(\omega) = H(\omega - \pi)$$

$$|H_0(\ )| = \left|\cos\left(\frac{\omega}{2}\right)\right| \quad ; \quad |H_1(\omega)| = \left|\sin\left(\frac{\omega}{2}\right)\right|$$



$$\cdot\; g_0[n] = (-1)^n h_1[n]$$

$$= (-1)^n (-1)^n h_0[n]$$

$$= \left((-1)^2\right)^n h_0[n]$$

$$= h_0[n] = h[n]$$

See PRHaarEg.m at web site

$$\cdot\; g_1[n] = -(-1)^n h_0[n]$$

$$= -(-1)^n h[n]$$

$h_0[n] = \left\{\frac{1}{2}, \frac{1}{2}\right\}$ $\quad$ $g_0[n] = \left\{\frac{1}{2}, \frac{1}{2}\right\}$

$h_1[n] = \left\{\frac{1}{2}, -\frac{1}{2}\right\}$ $\quad$ $g_1[n] = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$