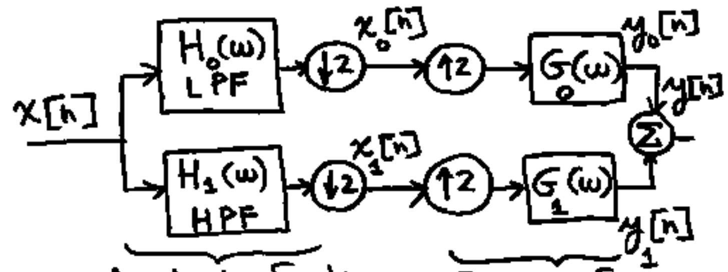
EE648 (CC761-M) DSP II
Session 20 (Live: 3/25/99)

Outline

- Perfect Reconstruction (PR) Filter Banks
  - Two-Channel Case
    - P+M Sect. 10.9.9
    - V - Chap. 5

- Analysis of Two-Channel PR Filter Bank



Desire: $y[n] = \alpha\, x[n-D]$

$$X_0(\omega) = \frac{1}{2} H_0\left(\frac{\omega}{2}\right) X\left(\frac{\omega}{2}\right) + \frac{1}{2} H_0\left(\frac{\omega-2\pi}{2}\right) X\left(\frac{\omega-2\pi}{2}\right)$$

$$X_1(\omega) = \frac{1}{2} H_1\left(\frac{\omega}{2}\right) X\left(\frac{\omega}{2}\right) + \frac{1}{2} H_1\left(\frac{\omega-2\pi}{2}\right) X\left(\frac{\omega-2\pi}{2}\right)$$

$$Y_0(\omega) = G_0(\omega)\, X_0(2\omega) = \frac{1}{2} G_0(\omega)\left\{H_0(\omega)X(\omega) + H_0(\omega-\pi)X(\omega-\pi)\right\}$$

$$Y_1(\omega) = \frac{1}{2} G_1(\omega)\left\{H_1(\omega)X(\omega) + H_1(\omega-\pi)X(\omega-\pi)\right\}$$

$$Y(\omega) = Y_0(\omega) + Y_1(\omega)$$

$$= \frac{1}{2}\left\{H_0(\omega)G_0(\omega) + H_1(\omega)G_1(\omega)\right\}X(\omega) + \underbrace{\frac{1}{2}\left\{H_0(\omega-\pi)G_0(\omega) + H_1(\omega-\pi)G_1(\omega)\right\}}_{\text{alinasing term}}X(\omega-\pi)$$

• to insure aliasing is zero, common practice (not only possibility, though) is to set:

$$G_0(\omega) = H_1(\omega-\pi) \Rightarrow g_0[n] = (-1)^n h_1[n]$$

$$G_1(\omega) = -H_0(\omega-\pi) \Rightarrow g_1[n] = -(-1)^n h_0[n]$$

In this case:

$$H_0(\omega-\pi)\,G_0(\omega) + H_1(\omega-\pi)\,G_1(\omega) = 0 \quad \text{for all } \omega$$

(Note: $e^{j\pi n} = (-1)^n$)

• at this point:

$$Y(\omega) = \frac{1}{2}\underbrace{\left\{ H_0(\omega) H_1(\omega-\pi) - H_1(\omega) H_0(\omega-\pi) \right\}}_{T(\omega) \Rightarrow \text{desire} = \alpha e^{-jD\omega}} X(\omega)$$

• let $h_0[n] \xleftrightarrow{DTFT} H_0(\omega)$ be a LPF of length N with symmetric impulse ($h_0[N-1-n] = h_0[n]$, $n = 0, 1, \dots, N-1$) response guaranteeing linear phase

• in the case of a Quadrature Mirror Filter (QMF) Bank:

$$h_1[n] = (-1)^n h_0[n]$$

$$\Rightarrow H_1(\omega) = H_0(\omega-\pi)$$

$\Rightarrow$ translates $h_0[n]$ into a HPF

• let $h[n]$ is a LPF - symmetric FIR with a cut-off near $\omega_c = \pi/2$

$$h_0[n] = h[n]$$

$$h_1[n] = (-1)^n h[n]$$

• Substituting

$$H_1(\omega) = H(\omega-\pi)$$

$$H_0(\omega) = H(\omega)$$

$$T(\omega) = H(\omega)\,H(\omega-2\pi) - H(\omega-\pi)\,H(\omega-\pi)$$

$$= H^2(\omega) - H^2(\omega-\pi)$$

• since $h[n]$ is symmetric:

$$H(\omega) = H_r(\omega)\, e^{-j\left(\frac{N-1}{2}\right)\omega}$$

$$H^2(\omega) = H_r^2(\omega)\, e^{-j(N-1)\omega}$$

$$H^2(\omega-\pi) = H_r^2(\omega-\pi)\, e^{-j(N-1)(\omega-\pi)}$$

$$= (-1)^{N-1} H_r^2(\omega-\pi)\; e^{-j(N-1)\omega}$$

- Substituting:

$$T(\omega) = \frac{1}{2}\left\{ |H(\omega)|^2 - (-1)^{N-1} |H(\omega-\pi)|^2 \right\} \cdot e^{-j(N-1)\omega}$$

$$= A(\omega)\, e^{-j(N-1)\omega}$$

- note: if N is odd, then $\omega = \pi/2$:

$$A(\omega)\Big|_{\omega=\frac{\pi}{2}} = |H(\tfrac{\pi}{2})|^2 - |H(\tfrac{\pi}{2})|^2 = 0$$

- Thus, choose N even! In this case:

$$A(\omega) = |H(\omega)|^2 + |H(\omega-\pi)|^2$$

- Ideally: $A(\omega) = 1$ for all ω

- For PR under these QMF conditions, we require a LPF $h[n]$ - symmetric FIR - with cut-off near $\omega_c = \pi/2$ satisfying the power-symmetry constraint

$$|H(\omega)|^2 + |H(\omega-\pi)|^2 = 1$$

What kind of filters satisfy this condition?

- Simplest Example:

$$h[n] = \left\{ \underset{\uparrow}{\tfrac{1}{2}}, \tfrac{1}{2} \right\} = \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-1]$$

$$H(\omega) = \frac{1}{2} + \frac{1}{2} e^{-j\omega}$$

$$= e^{-j\frac{\omega}{2}} \left\{ \frac{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}{2} \right\}$$

$$= \cos\left(\frac{\omega}{2}\right) e^{-j\frac{\omega}{2}}$$

$$|H(\omega)|^2 = \cos^2\left(\frac{\omega}{2}\right)$$

$$H(\omega-\pi) = \cos\left(\frac{\omega-\pi}{2}\right) e^{-j\frac{(\omega-\pi)}{2}}$$

$$= \cos\left(\frac{\omega}{2} - \frac{\pi}{2}\right) j\, e^{-j\frac{\omega}{2}}$$

$$= \sin\left(\frac{\omega}{2}\right) j\, e^{-j\frac{\omega}{2}}$$

$$|H(\omega)|^2 + |H(\omega-\pi)|^2$$

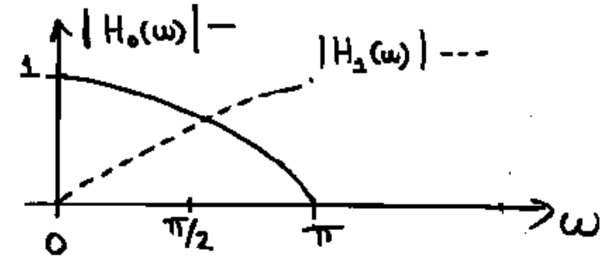
$$= \cos^2\left(\frac{\omega}{2}\right) + \sin^2\left(\frac{\omega}{2}\right) = 1 \quad \forall \omega$$

- satisfies power symmetry constraint!

$h_0[n] = h[n]$; $h_1[n] = (-1)^n h[n]$

$H_0(\omega) = H(\omega)$; $H_1(\omega) = H(\omega-\pi)$

$|H_0(\omega)| = |\cos(\frac{\omega}{2})|$; $|H_1(\omega)| = |\sin(\frac{\omega}{2})|$



- $g_0[n] = (-1)^n h_1[n]$

$$= (-1)^n (-1)^n h_0[n]$$

$$= ((-1)^2)^n h_0[n]$$

$$= h_0[n] = h[n]$$

- $g_1[n] = -(-1)^n h_0[n]$

$$= -(-1)^n h[n]$$

See PRHaarEg.m at web site

$h_0[n] = \{\frac{1}{2}, \frac{1}{2}\}$ $\quad g_0[n] = \{\frac{1}{2}, \frac{1}{2}\}$

$h_1[n] = \{\frac{1}{2}, -\frac{1}{2}\}$ $\quad g_1[n] = \{-\frac{1}{2}, \frac{1}{2}\}$