EE648 (CC761-M) DSP II

Session 23 Live: 4/6/99

Outline

- Perfect Reconstruction M-channel Filter Bank Employing Arbitrary FIR Analysis Filters of Length M
  - See Sect. V. 5.6.3

• Correction to notes for Session 21

$$h_{sr}[n] = \frac{2\beta \cos\left[(1+\beta)\pi \frac{(n+.5)}{2}\right]}{\pi\left[1-4\beta^2(n+.5)^2\right]}$$

$$+ \frac{\sin\left[(1-\beta)\pi \frac{(n+.5)}{2}\right]}{\pi(n+.5)\left[1-4\beta^2(n+.5)^2\right]}$$

$$-L \le n \le L-1$$

$$h[n] = h_{sr}[n-L], \quad n = 0, 1, \ldots, N-1$$

where: $N = 2L$

$h_0[n] = h[n]$; $g_0[n] = h[n]$

$h_1[n] = (-1)^n h[n]$; $g_1[n] = -h_1[n]$

$$Y(\omega) = \sum_{m=0}^{M-1} G_m(\omega) \sum_{l=0}^{M-1} H_m\left(\omega - l\frac{2\pi}{M}\right) X\left(\omega - l\frac{2\pi}{M}\right)$$

- In matrix form, define:

$$\underset{M \times 1}{\underline{G}(\omega)} = \begin{bmatrix} G_0(\omega) \\ G_1(\omega) \\ \vdots \\ G_{M-1}(\omega) \end{bmatrix} ; \quad \underline{X}(\omega) = \begin{bmatrix} X(\omega) \\ X\left(\omega - \frac{2\pi}{M}\right) \\ \vdots \\ X\left(\omega - (M-1)\frac{2\pi}{M}\right) \end{bmatrix}$$

$$\underset{M \times M}{\underline{\mathcal{H}}(\omega)} = \begin{bmatrix} H_0(\omega) & H_0\left(\omega - \frac{2\pi}{M}\right) & \cdots & H_0\left(\omega - (M-1)\frac{2\pi}{M}\right) \\ H_1(\omega) & H_1\left(\omega - \frac{2\pi}{M}\right) & \cdots & H_1\left(\omega - (M-1)\frac{2\pi}{M}\right) \\ \vdots & \vdots & \cdots & \vdots \\ H_{M-1}(\omega) & H_{M-1}\left(\omega - \frac{2\pi}{N}\right) & \cdots & H_{M-1}\left(\omega - (M-1)\frac{2\pi}{M}\right) \end{bmatrix}$$

$$Y(\omega) = \underbrace{\underline{G}^T(\omega)}_{1 \times M} \; \underbrace{\underline{\mathcal{H}}(\omega)}_{M \times M} \; \underbrace{\underline{X}(\omega)}_{M \times 1}$$

- PR can be achieved by enforcing

$$\underline{G}^T(\omega)\,\underline{H}(\omega) = [\alpha e^{-jD\omega}, 0, 0, \ldots, 0]$$

- <u>Goal</u>: show that for <u>arbitrary</u> analysis FIR filters of length M, PR can be achieved with synthesis filters of length M
  - M = no. of channels
    = decimation factor

$$\begin{bmatrix} G_0(\omega) \\ G_1(\omega) \\ \vdots \\ G_{M-1}(\omega) \end{bmatrix} = \underbrace{\begin{bmatrix} g_0[0] & g_0[1] & \cdots & g_0[M-1] \\ g_1[0] & g_1[1] & \cdots & g_1[M-1] \\ \vdots & \vdots & \vdots & \vdots \\ g_{M-1}[0] & g_{M-1}[1] & \cdots & g_{M-1}[M-1] \end{bmatrix}}_{} \underbrace{\begin{bmatrix} 1 \\ e^{-j\omega} \\ e^{-j2\omega} \\ \vdots \\ e^{-j(M-1)\omega} \end{bmatrix}}_{}$$

$$\underset{M\times 1}{\underline{G}(\omega)} = \underset{M\times M}{\underline{G}} \; \underset{M\times 1}{\underline{f}(\omega)}$$

• define:

$$\underset{M\times M}{\tilde{\underline{\underline{I}}}} = \begin{bmatrix} O & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 1 & & & O \end{bmatrix}$$

reverse permutation matrix

• e.g.

$$\underset{3\times 3}{\tilde{\underline{\underline{I}}}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

• <u>note</u>:

$$\tilde{\underline{\underline{I}}}\,\tilde{\underline{\underline{I}}} = \underline{\underline{I}} \qquad \tilde{\underline{\underline{I}}}^{-1} = \tilde{\underline{\underline{I}}}$$

$$\tilde{\underline{\underline{I}}}^{T} = \tilde{\underline{\underline{I}}}$$

$$\underline{\underline{\tilde{I}}}\,\underline{f}(\omega) = \left[e^{-j(M-1)\omega}, \ldots, e^{-j\omega}, \underline{1}\right]^T$$

$$= e^{-j(M-1)\omega}\left[1, e^{j\omega}, \ldots, e^{j(M-2)\omega}, e^{j(M-1)\omega}\right]^T$$

$$\Rightarrow \underline{\underline{\tilde{I}}}\,\underline{f}(\omega) = \underline{f}^*(\omega) \cdot e^{-j(M-1)\omega}$$

- $\left(\underline{\underline{\tilde{I}}}\,\underline{f}(\omega)\right)^T = \underline{f}^T(\omega)\,\underline{\underline{\tilde{I}}}^T = \underline{f}^T(\omega)\,\underline{\underline{\tilde{I}}}$

$$= e^{-j(M-1)\omega}\,\underline{f}^H(\omega)$$

- Next, rewrite $\underline{\mathcal{H}}(\omega)$
- recall: $e^{j\ell\frac{2\pi}{M}n} h_m[n] \overset{DTFT}{\longleftrightarrow} \hat{H}_m\left(\omega - \ell\frac{2\pi}{M}\right)$
- $\underline{\mathcal{H}}(\omega) = \underline{H}\ \underline{\Gamma}(\omega)$

$$\underline{H} = \begin{bmatrix} h_0[0] & h_0[1], & \cdots & h_0[M-1] \\ h_1[0] & h_1[1] & \cdots & h_1[M-1] \\ \vdots & \vdots & \vdots & \vdots \\ h_{M-1}[0] & h_{M-1}[1] & \cdots & h_{M-1}[M-1] \end{bmatrix} \quad M\times M$$

$$\underline{T}(\omega) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-j\omega} & e^{j\frac{2\pi}{M}}e^{-j\omega} & \cdots & e^{j(M-1)\frac{2\pi}{M}}e^{-j\omega} \\ e^{-j2\omega} & e^{j\frac{4\pi}{M}}e^{-j2\omega} & \cdots & e^{j(M-1)\frac{4\pi}{M}}e^{-j2\omega} \\ \vdots & \vdots & \cdots & \vdots \\ e^{-j(M-1)\omega} & e^{j(M-1)\frac{2\pi}{M}}e^{-j(M-1)\omega} & \cdots & e^{j(M-1)^2\frac{2\pi}{M}}e^{-j(M-1)\omega} \end{bmatrix}$$

$$Y(\omega) = e^{-j(M-1)\omega}\, \underline{f}^{H}(\omega) \left(\tilde{\underline{I}}\, \underline{G}^{T}\right) \underline{H}\, \underline{\Gamma}(\omega)\, \underline{X}(\omega)$$

- Suppose, we choose:

$$\tilde{\underline{I}}\, \underline{G}^{T} \underline{H} = \underline{I}$$

$$\tilde{\underline{I}}\, \underline{G}^{T} = \underline{H}^{-1}$$

$$\underline{G}^{T} = \tilde{\underline{I}}\, \underline{H}^{-1}$$

$$\Rightarrow \underline{G} = \left(\underline{H}^{-1}\right)^{T} \tilde{\underline{I}}$$

- With this $\underline{G}$, we have:

$$Y(\omega) = e^{-j(M-1)\omega} \, \underline{f}^H(\omega) \, \underline{T}(\omega) \, \underline{X}(\omega)$$

- ℓ-th element of $\underline{f}^H(\omega) \, \underline{T}(\omega)$ is

$$\sum_{n=0}^{M-1} e^{+jn\omega} \, e^{j 2\pi \frac{\ell}{M} n} \quad e^{-jn\omega}$$

$$= \sum_{n=0}^{M-1} e^{j 2\pi \frac{\ell}{M} n}$$

$$\sum_{n=0}^{M-1}\left(e^{j\frac{2\pi l}{M}}\right)^n = \frac{1-e^{j2\pi l}}{1-e^{j\frac{2\pi}{M}l}}$$

$$= \begin{cases} M, & l=0 \\ 0, & l=1,2,\ldots,M-1 \end{cases}$$

- **Thus:** $\underline{f}^H(\omega)\underline{T}(\omega) = [M, 0, 0, \ldots, 0]$

$$Y(\omega) = e^{-j(M-1)\omega} M\, X(\omega)$$

PR achieved with $\underline{G} = (\underline{H}^{-1})^T \underset{\sim}{\underline{I}}$

- See PRMchaneg.m

• **Example:** $\underline{H}$ = M pt. DFT matrix

row of 1s

$$\underline{H} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{M}} & e^{-j\frac{4\pi}{M}} & \cdots & e^{-j(M-1)\frac{2\pi}{M}} \\ 1 & e^{-j\frac{4\pi}{M}} & e^{-j\frac{8\pi}{M}} & \cdots & e^{-j(M-1)\frac{4\pi}{M}} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{-j(M-1)\frac{2\pi}{M}} & e^{-j(M-1)\frac{4\pi}{M}} & \cdots & e^{-j(M-1)^2\frac{2\pi}{M}} \end{bmatrix}$$

• Note: $\underline{H}\,\underline{H}^H = M\,\underline{I}$

Thus: $\underline{H}^{-1} = \frac{1}{M}\underline{H}^H$

$$\underline{G} = \left(\underline{H}^{-1}\right)^T \underline{\tilde{I}}$$

$$= \left(\frac{1}{M}\,\underline{H}^H\right)^T \underline{\tilde{I}}$$

$$= \frac{1}{M}\,\underline{H}^* \underline{\tilde{I}}$$

• post-multiplying by $\underline{\tilde{I}}$ reverses the order of each row of $\underline{H}^*$

$$\begin{array}{c} 1, \quad 1, \quad \cdots \quad 1 \;\} \text{ row of 1's} \end{array}$$

$$\underline{H}^*\underline{\tilde{I}} = \begin{bmatrix} e^{+j(M-1)\frac{2\pi}{M}}, & e^{j(M-2)\frac{2\pi}{M}}, & \cdots, & e^{j\frac{2\pi}{M}}, & 1 \\ e^{j(M-1)\frac{4\pi}{M}}, & e^{j(M-2)\frac{4\pi}{M}}, & \cdots, & e^{j\frac{4\pi}{M}}, & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ e^{j(M-1)^2\frac{2\pi}{M}}, & e^{j(M-1)(M-2)\frac{2\pi}{M}}, & \cdots, & e^{j(M-1)\frac{2\pi}{M}}, & 1 \end{bmatrix}$$

$$\underline{G} = \underline{H}^{*}\underline{\tilde{I}} = \begin{array}{l} \left.\begin{matrix} 1 & 1 & \cdots & 1 \end{matrix}\right\} \text{row of one's} \\ \end{array}$$

$$= \begin{bmatrix} e^{+j(M-1)\frac{2\pi}{M}}\left\{1,\ e^{-j\frac{2\pi}{M}},\ \cdots,\ e^{-j(M-1)\frac{2\pi}{M}}\right\} \\ e^{+j(M-1)\frac{4\pi}{M}}\left\{1,\ e^{-j\frac{4\pi}{M}},\ \cdots,\ e^{-j(M-1)\frac{4\pi}{M}}\right\} \\ \vdots \\ e^{+j(M-1)^2\frac{2\pi}{M}}\left\{1,\ e^{-j(M-1)\frac{2\pi}{M}},\ \cdots,\ e^{-j(M-1)^2\frac{2\pi}{M}}\right\} \end{bmatrix}$$

note: $e^{j(M-1)\ell\frac{2\pi}{M}} = e^{j\ell 2\pi}\, e^{-j\ell\frac{2\pi}{M}}$

$\ell = 0, 1, \cdots, M-1$

- Thus:

$$\underline{G} = \underline{\Phi}\ \underline{H}$$

$$\underline{\Phi} = \begin{bmatrix} 1 & & & 0 \\ & e^{-j\frac{2\pi}{N}} & & \\ & & e^{-j\frac{4\pi}{N}} & \\ 0 & & & \ddots \\ & & & & e^{-j(N-1)\frac{2\pi}{N}} \end{bmatrix}$$

$$g_m[n] = e^{-j\frac{2\pi}{N}m}\ h_m[n] \quad , \quad \begin{array}{l} n = 0, 1, \ldots, N-1 \\ m = 0, 1, \ldots, N-1 \end{array}$$

See PRDFTeg.m and PRMchaneg.m