EE648 (CC761-M) DSP II

Session 28 (Live: 4/22/99)

<u>Outline</u>

- Continuous-Time Orthonormal Wavelet Basis
  - Sect. 11.5 of V. Text

- $x(t) = \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} X_{DWT}(k,n)\, \psi_{kn}(t)$

- $\psi_{kn}(t)$: dilations $(t \to a^{-k} t)$

  and shifts $(t \to t - n a^{k} T)$

  of "mother" wavelet $\psi(t)$

- If $\psi_{kn}(t)$ form a <u>complete</u> orthonormal basis, i.e.,

$$\int_{-\infty}^{\infty} \psi_{kn}(t)\, \psi_{\ell m}^{*}(t)\, dt$$

$$= \delta[\underbrace{k-\ell}_{\text{subband index}}]\;\delta[\underbrace{n-m}_{\text{sample/time/shift index}}]$$

- then:

$$X_{DWT}(k,n) = \int_{-\infty}^{\infty} x(t)\, \psi_{kn}^{*}(t)\, dt$$

- recall from Session 27:

$$X_{DWT}(k,n) = \int_{-\infty}^{\infty} x(t)\, h_k(\tau - t)\, dt \Big|_{\tau = na^k T}$$

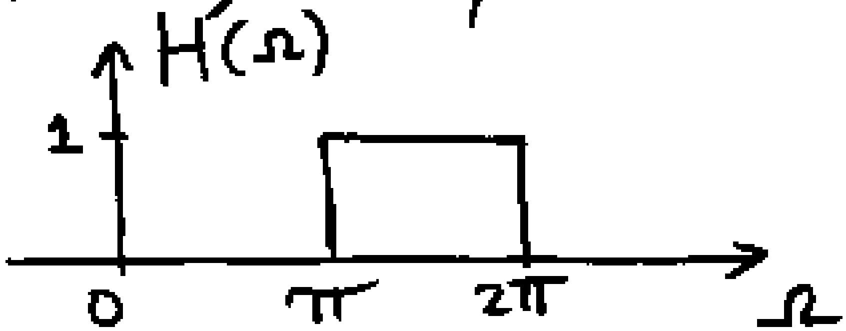
$$= \int_{-\infty}^{\infty} x(t)\, a^{-\frac{k}{2}}\, h\left(a^{-k}(na^k T - t)\right) dt$$

- $\psi_{kn}(t) = a^{-\frac{k}{2}}\, h^*\left(a^{-k}(na^k T - t)\right)$
- so that $\psi_{00}(t) = \psi(t) = h^*(-t)$

$$\Rightarrow \Psi(\Omega) = H^*(\Omega)$$

- here concentrate on case of $a=2$ corresponding to diadic or binary wavelet decomposition
- also WLOG we will set $T = 1$ sec.
- SUBSTANTIATION:
- first notes the subband output requiring the highest sampling rate is that passed by $H_0(\Omega) = H(\Omega)$ $(= \Psi^*(\Omega))$
- the passband width of $H(\Omega)$ can be at most $2\pi \frac{1}{2} 1 = \pi \ \frac{\text{rads.}}{\text{sec.}}$

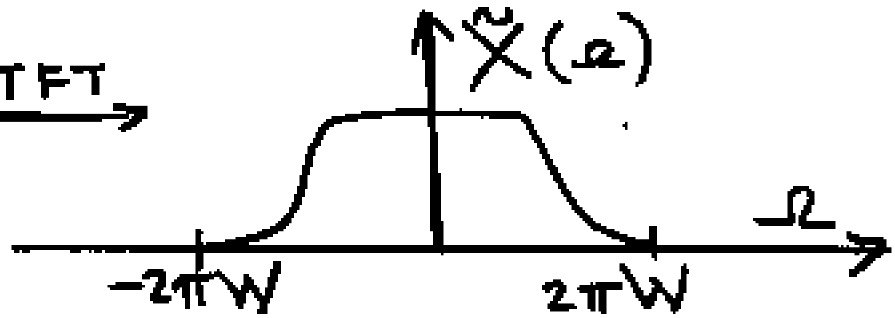
• In fact, ideally:



• recall: $x(at) \xleftrightarrow{\text{CTFT}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$

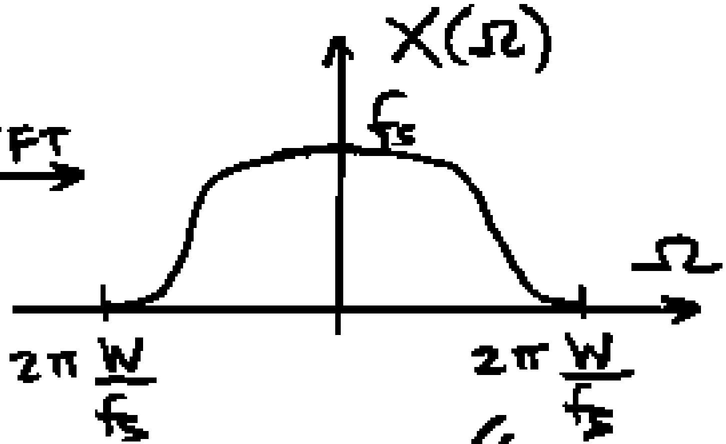
• thus, if

$\tilde{x}(t) \xleftrightarrow{\text{CTFT}}$



• then

$$x(t) = \tilde{x}\left(\frac{t}{f_s}\right) \xleftrightarrow{CTFT}$$



• can choose $f_s = W$ so that $\underset{2\pi}{''}$

• thus, expanding $x(t)$ as

$$\bullet\; x(t) = \sum_k \sum_n X_{DWT}(k,n)\,\psi\left(2^{-k}\left(t - n2^{k}\right)\right)$$

• then:

$$\tilde{x}(t) = x(f_s t) = \sum_k \sum_n X_{DWT}(k,n)\,\psi\left(2^{-k}\left(f_s t - n2^{k}\right)\right)$$

- Consider forming $\Psi(\Omega)$ $\left(= H^*(\Omega)\right)$
- Wavelet Function:

$$\Psi(\Omega) = \frac{1}{\sqrt{2}} H_1^{(2)}(2^{-1}\Omega) \prod_{m=2}^{\infty} H_0^{(2)}(2^{-m}\Omega)$$

- where: $H_0^{(2)}(2^{-m}\Omega) = H_0^{(2)}(\omega)\Big|_{\omega = 2^{-m}\Omega}$

$$H_1^{(2)}(2^{-m}\Omega) = H_1^{(2)}(\omega)\Big|_{\omega = 2^{-m}\Omega}$$

- See Fig. 11.5-1 in V. Text
- Note: $\Psi(\Omega)$ is <u>not</u> periodic

• need to also define a complementary function (which is lowpass) referred to as the <u>scale function</u> $\phi(t)$ with CTFT $\Phi(\Omega)$ formed as

$$\Phi(\Omega) = \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-1}\Omega) \prod_{m=2}^{\infty} H_0^{(2)}(2^{-m}\Omega)$$

• can show:

$$\Psi(\Omega) = \frac{1}{\sqrt{2}} H_1^{(2)}(2^{-1}\Omega)\, \Phi(2^{-1}\Omega)$$

$$\Phi(\Omega) = \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-1}\Omega)\, \Phi(2^{-1}\Omega)$$

• VIP Points :

1. If $H_0^{(2)}(\omega)$ satisfies power symmetry and $H_0^{(2)}(\omega) \neq 0$ for $|\omega| < \frac{\pi}{2}$ $\Rightarrow$

$$\psi_{kn}(t) = 2^{-k/2}\,\psi\left(2^{-k}(t - n2^{k})\right)$$

$k = 0, 1, \ldots, \infty \qquad n = -\infty, \ldots, \infty$

form a <u>complete</u>, othonormal basis for $L^2(\mathbb{R})$ - set of all finite energy functions defined as a fn. of single real-valued independent variable

2. Can make $\psi(t)$ "smooth" or "regular" by constraining $H_0^{(2)}(\omega)$ to have multiple zeroes at $\omega = \pi$ ($H_0^{(2)}(z)$ has multiple zeroes at $z = -1$)

3. If $h_0^{(2)}[n]$ (and $h_1^{(2)}[n] = (-1)^n h_0^{(2)}[n]$) are FIR, $\psi_{k,n}(t) = 2^{-k}\psi\left(2^{-k}(t - n2^{k})\right)$ are all of finite duration

- First need to study infinite products
- Case I. $S = \prod_{m=1}^{\infty} a^{b^m} = a^{\sum_{m=1}^{\infty} b^m}$

$$= a^{\frac{1}{1-b} - 1} = a^{\frac{b}{1-b}} \qquad |b| < 1$$

- Case II. $\prod_{m=1}^{\infty} \cos(2^{-m}\Omega) = ?$

Trick: $\sin(2\alpha) = 2 \sin\alpha \cos\alpha$

$$\cos\alpha = \frac{\sin 2\alpha}{2 \sin\alpha}$$

$$\prod_{m=1}^{K} \cos(2^{-m}\Omega) = \prod_{m=1}^{K} \frac{\sin\left(2^{-(m-1)}\Omega\right)}{2\sin(2^{-m}\Omega)}$$

- den. of m-th term cancels the numerator of the $(m+1)$-th term

$$\lim_{K\to\infty} \prod_{m=1}^{K} \cos(2^{-m}\Omega) = \frac{\sin\Omega}{\Omega}$$

- since for large $K \Rightarrow \sin(2^{-K}\Omega) = 2^{-K}\Omega$