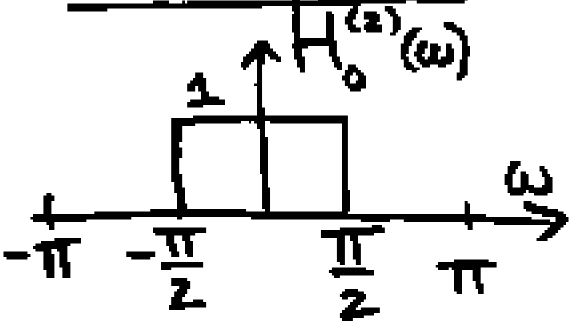
EE648 (CC761-M) DSP II

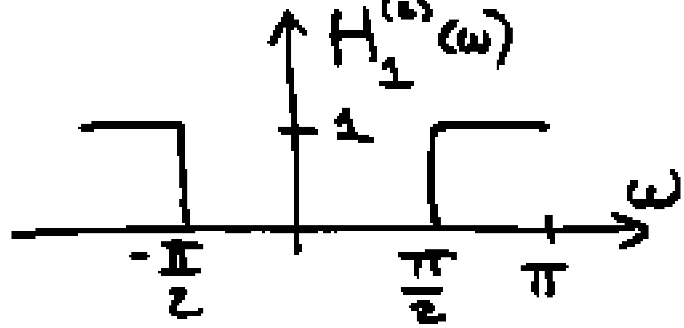
Session 29 (Live: 4/24/99)

Outline

- Continuous-Time Orthonormal Wavelet Basis (cont.)
  - Sect. 11.5 of V. Text

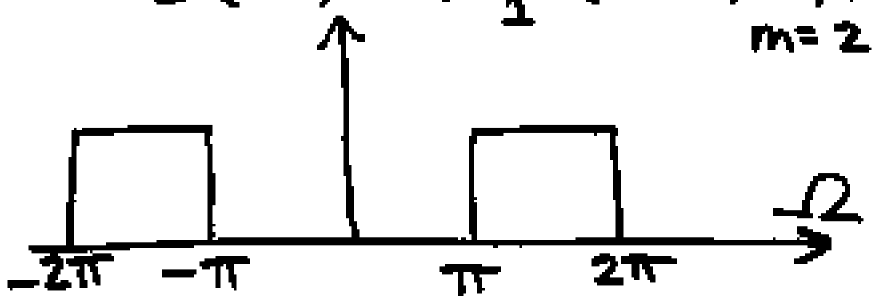
• **Example 1.** Ideal half-band filters





• See Fig. 11-5.1

$$\Psi(\Omega) = H_1^{(2)}(2^{-1}\Omega) \prod_{m=2}^{\infty} H_0^{(2)}(2^{-m}\Omega)$$

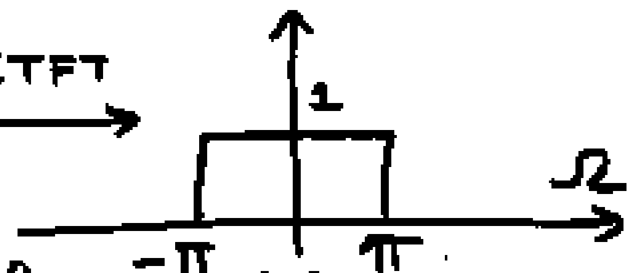


- in this case, the "mother" wavelet is

$$\psi(t) = \frac{\sin\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}t} \cdot \cos\left(\frac{3}{2}\pi t\right)$$

- Corresponding scale function $\phi(t)$

$$\phi(t) = \frac{\sin(\pi t)}{\pi t} \quad \overset{\text{CTFT}}{\longleftrightarrow}$$



- $\psi_{kn}(t) = 2^{k}\,\psi\left(2^{-k}(t - n2^{k})\right)$

$k = 0, 1, \ldots, \infty$, $n = -\infty, \ldots, \infty$ $\Rightarrow$ form complete orthonormal basis for $L^2(R)$

• Example 2. Haar Wavelet Basis

$$h_0^{(2)}[n] = \frac{1}{\sqrt{2}}\left\{\delta[n] + \delta[n-1]\right\}; \quad h_1^{(2)}[n] = \frac{1}{\sqrt{2}}\left\{\begin{array}{l}\delta[n] \\ -\delta[n-1]\end{array}\right\}$$

$$H_0^{(2)}(\omega) = \frac{1}{\sqrt{2}}\left\{1 + e^{-j\omega}\right\}; \quad H_1^{(2)}(\omega) = \frac{1}{\sqrt{2}}\left\{1 - e^{-j\omega}\right\}$$

$$= \sqrt{2}\, e^{-j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right) \qquad ; = \sqrt{2}\, j\, e^{-j\frac{\omega}{2}} \sin\left(\frac{\omega}{2}\right)$$

$$\Phi(\Omega) = \prod_{m=1}^{\infty} \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-m}\Omega)$$

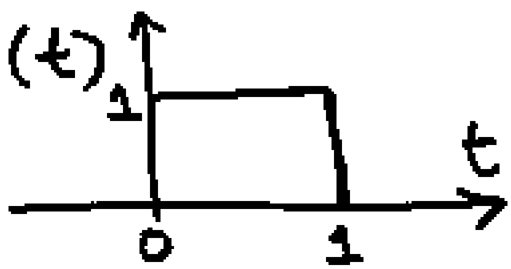
$$= \prod_{m=1}^{\infty} \frac{1}{\sqrt{2}} \sqrt{2}\, e^{-j\frac{2^{-m}\Omega}{2}} \cos\left(\frac{2^{-m}\Omega}{2}\right)$$

$$= \prod_{m=1}^{\infty} e^{-j 2^{-m}\frac{\Omega}{2}} \prod_{m=1}^{\infty} \cos\left(2^{-m}\frac{\Omega}{2}\right)$$

$$e^{-j\frac{\Omega}{2}\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m} \cdot \frac{\sin\left(\frac{\Omega}{2}\right)}{\Omega/2}$$

$$= e^{-j\frac{\Omega}{2}} \frac{\sin\left(\frac{\Omega}{2}\right)}{\Omega/2} = \Phi(\Omega)$$

$\phi(t)$



$$\phi(t) = \mathrm{rect}\left(\frac{t-.5}{1}\right)$$

$$= \mathrm{rect}(t-.5)$$

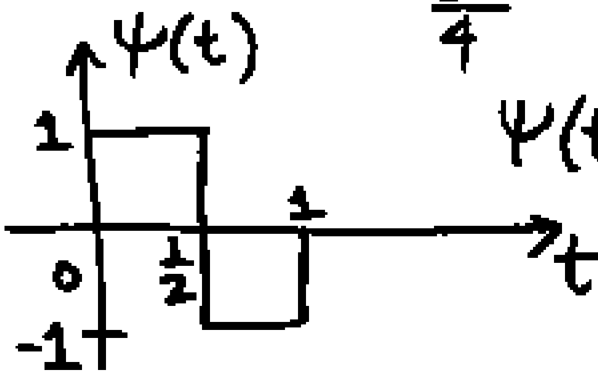
• Next, compute CTFT of "mother" wavelet, $\psi(t)$

$$\Psi(\Omega) = \frac{\Phi(\Omega)}{\frac{1}{\sqrt{2}} H_0^{(2)}(2^{-1}\Omega)} \cdot \frac{1}{\sqrt{2}} H_1^{(2)}(2^{-1}\Omega)$$

$$= \frac{e^{-j\frac{\Omega}{2}} \dfrac{\sin\left(\frac{\Omega}{2}\right)}{\frac{\Omega}{2}}}{\sqrt{2}\, e^{-j\frac{\Omega}{4}} \cos\left(\frac{\Omega}{4}\right)} \cdot \sqrt{2}\, e^{-j\frac{\Omega}{4}} \sin\left(\frac{\Omega}{4}\right)$$

$$= \frac{e^{-j\frac{\Omega}{2}}}{\frac{\Omega}{2}} \cdot \frac{2\sin\left(\frac{\Omega}{4}\right)\cos\left(\frac{\Omega}{4}\right)\ \sin\left(\frac{\Omega}{4}\right)}{\cos\left(\frac{\Omega}{4}\right)}$$

$$= e^{-j\frac{\Omega}{2}}\ \frac{\sin\left(\frac{\Omega}{4}\right)}{\frac{\Omega}{4}} \cdot \sin\left(\frac{\Omega}{4}\right) = \Psi(\Omega)$$



$$\psi(t) = \mathrm{rect}\left(\frac{t - 1/4}{1/2}\right) - \mathrm{rect}\left(\frac{t - 3/4}{1/2}\right)$$

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- The Haar Basis forms a complete orthonormal basis for any function in $L^2(\mathbb{R})$

$$x(t) = \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} X_{DWT}(k,n) \underbrace{2^{-k/2}\,\psi\left(2^{-k}(t-n2^{k})\right)}$$

$$\psi_{kn}(t) = \psi\left(2^{-k}t - n\right)$$

$$\psi_{00}(t) = \psi(t)$$

$$\psi_{01}(t) = \psi(t-1)$$

$$\psi_{10}(t) = \frac{1}{\sqrt{2}}\,\psi\left(t/2\right)$$

$$\psi_{11}(t) = \frac{1}{\sqrt{2}}\,\psi\left(\tfrac{1}{2}(t-2)\right)$$

See Fig. 11.5-7 in V. Text on pg. 521 ⇒ depicts orthogonality

- Basic Functions with Finite Duration
- desire to show that if $h_0^{(2)}[n]$ and $h_1^{(2)}[n]$ are FIR, $\psi(t), \phi(t)$ have finite duration
- Mathematical Preliminaries

$$\delta_a(t-n) \xleftrightarrow{CTFT} e^{-j\Omega n}$$

$$\sum_{n=0}^{N-1} h_0^{(2)}[n]\,\delta_a(t-n) \xleftrightarrow{CTFT} H_0^{(2)}(\Omega)$$

$$= \sum_{n=0}^{N-1} h_0[n]\, e^{-j\Omega n} = H_0^{(2)}(\omega)\Big|_{\omega=\Omega}$$

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- recall scaling property of CTFT

$$g(at) \xleftrightarrow{CTFT} \frac{1}{|a|} G\left(\frac{\Omega}{a}\right)$$

Here $a = 2^{+m}$

$$\sum_{n=0}^{N-1} h_0^{(2)}[n] \frac{2^m}{\sqrt{2}} \delta_a(2^m t - n) \xleftrightarrow{CTFT} \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-m}\Omega)$$

$$\phi(t) = \frac{2}{\sqrt{2}} \sum_{n=0}^{N-1} h_0^{(2)}[n] \delta_a(2t-n)$$

$$* \frac{4}{\sqrt{2}} \sum_{n=0}^{N-1} h_0^{(2)}[n] \delta_a(4t-n)$$

$$* \frac{8}{\sqrt{2}} \sum_{n=0}^{N-1} h_0^{(2)}[n] \delta_a(8t-n)$$

$$* \cdots$$

Time Duration:

0 to $\frac{N}{2}$

0 to $\frac{N}{4}$

0 to $\frac{N}{8}$

$\vdots$

$$\phi(t) \stackrel{CTFT}{\longleftrightarrow} \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-1}\Omega) \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-2}\Omega) \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-3}\Omega) \cdots$$

- overall time duration:

$$\frac{N}{2} + \frac{N}{4} + \frac{N}{8} + \cdots = N \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m$$

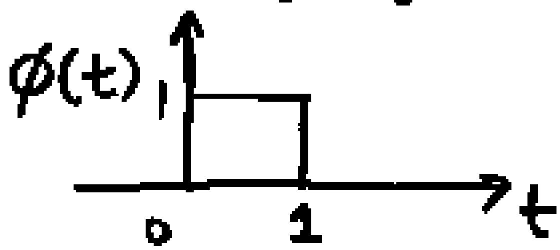
$$= N \left\{ \frac{1}{1-\frac{1}{2}} - 1 \right\} = N$$

- can similarly show that $\psi(t)$ is of finite duration of N secs.

- Example. Haar Basis

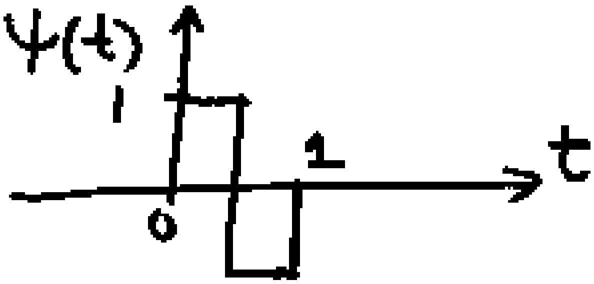
$$h_0^{(2)}[n] = \frac{1}{\sqrt{2}}\left\{\delta[n] + \delta[n-1]\right\}$$

$$h_1^{(2)}[n] = \frac{1}{\sqrt{2}}\left\{\delta[n] - \delta[n-1]\right\}$$



duration:

N=1 sec



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- Self-Similarity:

$$\underbrace{\Phi(\Omega) = \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-1}\Omega) \prod_{m=2}^{\infty} \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-m}\Omega)}$$

$$\underbrace{\prod_{m=2}^{\infty} \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-m}\Omega)}_{\Phi\left(\frac{\Omega}{2}\right)}$$

$$\phi(t) = \frac{1}{\sqrt{2}} \left\{ 2 \sum_n h_0^{(2)}[n]\, \delta_a(2t-n) \right\} * \{2\,\phi(2t)\}$$

$$\phi(t) = \left(\overbrace{\frac{2}{\sqrt{2}}}^{\sqrt{2}}\right) \sum_{n=0}^{N-1} h_0^{(2)}[n]\, \phi\left(2\left(t - \frac{n}{2}\right)\right)$$

- since $\delta_a(2t-n) = \delta_a\left(2\left(t-\frac{n}{2}\right)\right)$

$$= \frac{1}{2}\, \delta_a\left(t - \frac{n}{2}\right)$$

$$\phi(t) = \sqrt{2} \sum_{n=0}^{N-1} h_0^{(2)}[n]\, \phi(2t-n)$$

= superposition of weighted and shifted versions of $\phi(2t)$ (since $\phi(2t-n) = \phi\left(2\left(t-\frac{n}{2}\right)\right)$)

- See Fig. 11.5-5 in Y. Text