

EE648 (CC761-M) DSP II
Session 3 (hive: 1/19/99)

Outline:

- Further analysis of convergence of LMS
- Sect. 12.2.3 of 1st Ed. of P+M
- Application: Adaptive Noise Cancellation - Sect. 12.1 of 1st Ed. P+M

• recall LMS update eqn.:

$$\underline{h}_M[n+1] = \underline{h}_M[n] + \mu e[n] \underline{x}[n]$$

$$\underline{h}_M[n+1] - \underline{h}_M^{\text{opt}} = \underline{h}_M[n] - \underline{h}_M^{\text{opt}} + \mu e[n] \underline{x}[n]$$

• take expected value of both sides:

$$\underline{e}[n+1] = \underline{e}[n] + \mu E \left\{ \left[d[n] - \underline{x}^T[n] \underline{h}_M[n] \right] \underline{x}[n] \right\}$$

• where: $\underline{s}[n] = E \left\{ \underline{h}_M[n] - \underline{h}_M^{\text{opt}} \right\}$

$$\begin{aligned}
 \epsilon[n+1] &= \epsilon[n] + \mu \Gamma_{dx} \\
 &\quad - \underbrace{\mu E\{x[n]x^T[n]h_M[n]\}} \\
 &\quad - \mu E\{x[n]x^T[n](h_M[n] - h_M^{opt})\} \\
 &\quad - \mu E\{x[n]x^T[n]\}h_M^{opt}
 \end{aligned}$$

• See: "Adaptive Signal Processing"
 by Widrow & Stearns
 Prentice-Hall, 1985, Chap. 6

• Widrow proved error

$$\underline{h}_M[n] - \underline{h}_M^{\text{opt}}$$

is uncorrelated with data $\underline{x}[n]$. Thus:

$$\begin{aligned} & E \left\{ \underline{x}[n] \underline{x}^T[n] (\underline{h}_M[n] - \underline{h}_M^{\text{opt}}) \right\} \\ &= E \left\{ \underline{x}[n] \underline{x}^T[n] \right\} E \left\{ \underline{h}_M[n] - \underline{h}_M^{\text{opt}} \right\} \\ &= \underline{R}_{xx} \underline{e}[n] \end{aligned}$$

• in addition:

$$- \mu E \{ \underline{x}[n] \underline{x}^T[n] \} \underline{h}_M^{opt}$$

$$= - \mu \underline{R}_{xx} (\underline{R}_{xx}^{-1} \underline{r}_{dx})$$

$$= - \mu \underline{r}_{dx}$$

• Ultimately, after substitution:

$$\underline{c}[n+1] = \underline{c}[n] - \mu \underline{R}_{xx} \underline{c}[n]$$

$$= \left\{ \underline{I}_M - \mu \underline{R}_{xx} \right\} \underline{c}[n]$$

• consider eigenvalue decomposition
of \underline{R}_{xx} : $\underline{R}_{xx} = \underline{U} \underline{\Lambda} \underline{U}^T$

• Since \underline{R}_{xx} is symmetric

$$\underline{U}^T \underline{U} = \underline{I}_M = \underline{U} \underline{U}^T$$

$$\underline{c}[n+1] = \{ \underline{U} \underline{U}^T - \mu \underline{U} \underline{\Lambda} \underline{U}^T \} \underline{c}[n]$$

$$\underline{U}^T \underline{c}[n+1] = \{ \underline{I}_M - \mu \underline{\Lambda} \} \underline{U}^T \underline{c}[n]$$

• define: $\underline{c}^\circ[n] = \underline{U}^T \underline{c}[n]$

$$\underline{c}^\circ[n+1] = \left\{ \underline{I}_n - \mu \underline{A} \right\} \underline{c}^\circ[n]$$

• component-wise:

$$c^\circ[k; n+1] = (1 - \mu \lambda_k) c^\circ[k; n]$$

• recall: $h[k; n] = a h[k; n-1]$ $k=1, \dots, M$

$$\Rightarrow \text{sol}^n: h[k; n] = a^n h[k; 0]$$

• thus: $c^\circ[k; n] = (1 - \mu \lambda_k)^n c^\circ[k; 0]$

• for convergence, require:

$$-1 < 1 - \mu \lambda_k < 1 \quad \text{for } k=1, \dots, M$$

$$1 > \mu \lambda_k - 1 > -1$$

$$-1 < \mu \lambda_k - 1 < 1$$

$$0 < \mu \lambda_k < 2$$

$$0 < \mu < \frac{2}{\lambda_k}$$

λ_k are
strictly
non-negative

• to insure convergence:

$$0 < \mu < \frac{2}{\lambda_{\max}}$$

• to be conservative : $\mu < \frac{1}{\lambda_{\max}}$

• in practice:

$$\lambda_{\max} < \sum_{k=1}^M \lambda_k = \text{trace}\{R_{xx}\} \\ = M r_{xx}[0]$$

• thus:

$$0 < \mu < \frac{1}{M r_{xx}[0]} \\ \text{or } \left(\frac{2}{M r_{xx}[0]} \right)$$

• further analysis :

• say, we choose: $\mu = \frac{1}{\lambda_{\max}}$

$$c^{\circ}[k; n] = \left(1 - \frac{\lambda_{k}}{\lambda_{\max}}\right)^n c^{\circ}[k; 0]$$

$k = 1, \dots, M$

• consider $k = M$, for
which $\lambda_M = \lambda_{\min}$

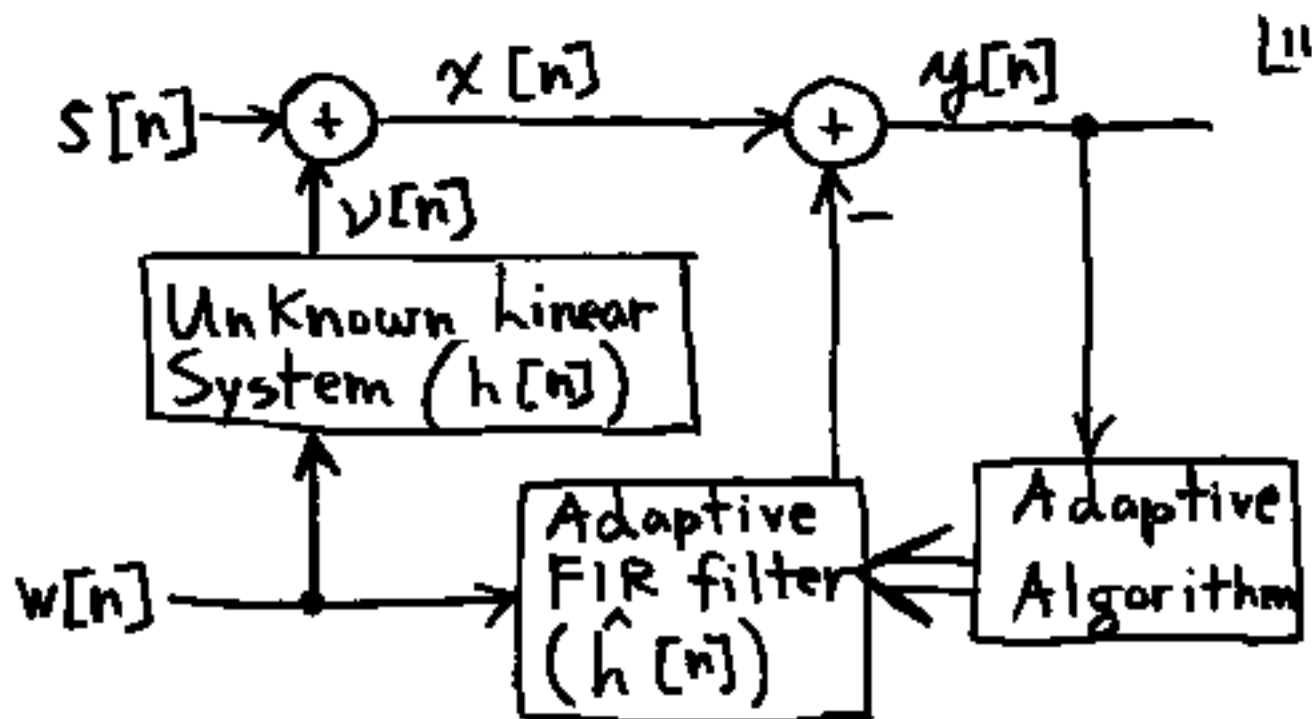
(assuming $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M$)

• since: $1 - \frac{\lambda_{\min}}{\lambda_{\max}} > 1 - \frac{\lambda_k}{\lambda_{\max}}$

- so the M -th term associated with the smallest eigenvalue, λ_{\min} , takes the longest to converge
 - $\frac{\lambda_{\max}}{\lambda_{\min}}$ determines the convergence rate of LMS
- the Recursive Least Squares (RLS) algorithm to be developed in Session 4 is not as sensitive to the eigenvalue spread of R_{xx}

- Application: Adaptive Noise Cancellation
- specific problem:
- car phone imbedded in steering wheel (or wearing a head set)
- speech is potentially masked by "car noise"
- approach: place a microphone at some point in car to pick up "car noise" only (negligible speech)
- exploit correlation between noise

picked up at "remote mike" and ¹¹⁰
noise contaminating speech
to form an estimate of the latter
and subtract it off



$x[n]$: plays the role of the "desired" signal, $d[n]$
 $y[n]$: plays the role of the "error" signal, $e[n]$

• $S[n]$: speech-only (unobservable)

• $w[n]$: noise-only observed at "remote" sensor (mike)

• $v[n]$: filtered version of $w[n]$ that corrupts speech signal
$$v[n] = w[n] * h[n]$$

• $h[n]$: FIR filter of length M

• $x[n] = S[n] + v[n] \Rightarrow$ picked up by transmitting mike

Assumption: $w[n]$ and $S[n]$ are independent random processes

Choose $\hat{h}[n]$ to minimize

$$E \left\{ \left[x[n] - \sum_{k=0}^{M-1} \hat{h}[n] w[n-k] \right]^2 \right\}$$

$$= E \left\{ \left[x[n] - \hat{\underline{h}}_M^T \underline{w}[n] \right]^2 \right\}$$

where:

$$\hat{\underline{h}}_M = [\hat{h}[0], \dots, \hat{h}[M-1]]^T$$

$$\underline{w}[n] = [w[n], \dots, w[n-(M-1)]]^T$$

$$\begin{aligned} \nabla_{\hat{\underline{h}}_M} E\{y^2[n]\} &= \\ \nabla_{\hat{\underline{h}}_M} \left\{ E\{x^2[n]\} - 2 \hat{\underline{h}}_M^T E\{x[n] \underline{w}[n]\} \right. \\ &\quad \left. + \hat{\underline{h}}_M^T E\{\underline{w}[n] \underline{w}^T[n]\} \hat{\underline{h}}_M \right\} \\ &= -2 \underline{\Gamma}_{wx} + 2 \underline{R}_{ww} \hat{\underline{h}}_M = \underline{0} \end{aligned}$$

$$\underline{R}_{ww} = E\{\underline{w}[n] \underline{w}^T[n]\}$$

$$\underline{\Gamma}_{wx} = E\{x[n] \underline{w}[n]\}$$

• sol'n: $\hat{\underline{h}}_M^{\text{opt}} = \underline{R}^{-1} \underline{r}_{wx}$

• show: $\hat{\underline{h}}_M^{\text{opt}} = \underline{h}$

• recall: $x[n] = s[n] + v[n]$
 $= s[n] + w[n] * h[n]$
 $= s[n] + \underline{h}^T \underline{w}[n]$

$E\{y^2[n]\}$
 $= E\{[s[n] + \underline{h}^T \underline{w}[n] - \hat{\underline{h}}_M^T \underline{w}[n]]^2\}$

$$\begin{aligned}
 &= E\left\{ \left[s[n] + (\underline{h} - \hat{\underline{h}}_M)^T \underline{w}[n] \right]^2 \right\} \\
 &= E\left\{ s^2[n] \right\} + 2(\underline{h} - \hat{\underline{h}}_M)^T E\left\{ s[n] \underline{w}[n] \right\} \\
 &\quad + (\underline{h} - \hat{\underline{h}}_M)^T E\left\{ \underline{w}[n] \underline{w}^T[n] \right\} (\underline{h} - \hat{\underline{h}}_M) \\
 &= E\left\{ s^2[n] \right\} + (\underline{h} - \hat{\underline{h}}_M)^T \underline{R}_{ww} (\underline{h} - \hat{\underline{h}}_M)
 \end{aligned}$$

• thus: $\hat{\underline{h}}_M^{\text{opt}} = \underline{h}$ since \underline{R}_{ww} is positive-definite