EE648 (CC761-M) DSP II

Session 30 (Live: 4/29/99)

Outline

- Final Comment on Wavelets
  - Sect. 11-5. of V. Text

• Motivation for forming the CTFT of scale fn. $\phi(t)$ and the mother wavelet $\psi(t)$ via infinite products:

$$\Phi(\Omega) = \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-1}\omega) \prod_{m=2}^{\infty} \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-m}\omega) \Bigg|_{\omega=\Omega}$$

$$\Psi(\Omega) = \frac{1}{\sqrt{2}} H_1^{(2)}(2^{-1}\omega) \prod_{m=2}^{\infty} \frac{1}{\sqrt{2}} H_0^{(2)}(2^{-m}\omega) \Bigg|_{\omega=\Omega}$$

• where: $H_0^{(2)}(\omega)$ and $H_1^{(2)}(\omega)$ comprise PR 2-channel filter bank

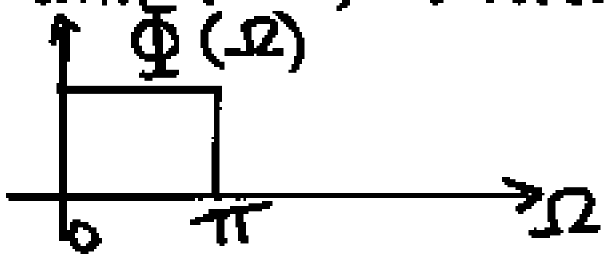
• Recall: $$x(t) = \sum_{k=1}^{\infty} \sum_{n=-\infty}^{\infty} \psi_{DWT}(k,n)\, \psi_{kn}(t)$$

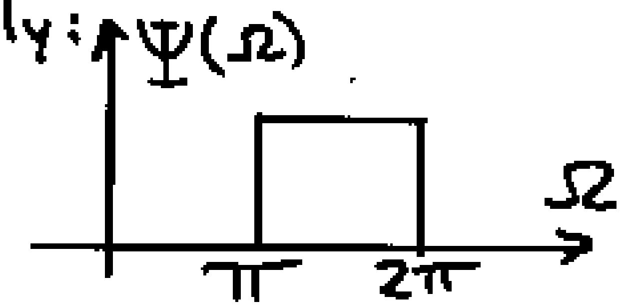
$$\psi_{kn}(t) = 2^{-k/2}\, \psi\left(2^{-k}(t - n2^{k}T)\right)$$

• where: $$\psi_{DWT}(k,n) = \int_{-\infty}^{\infty} x(t)\, \psi_{kn}^{*}(t)\, dt$$

• where $T = 1$ WLOG (see Session 28)

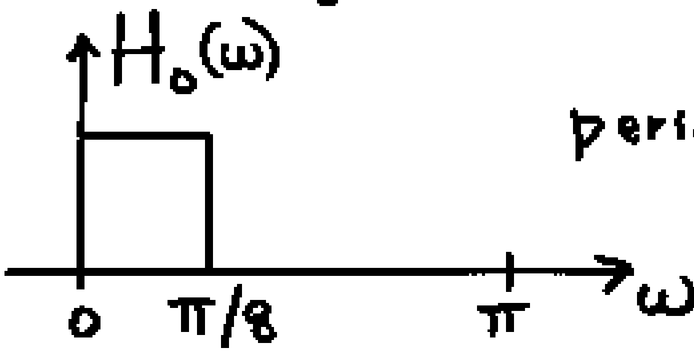
• with $T = 1$, $\Rightarrow$ ideally:





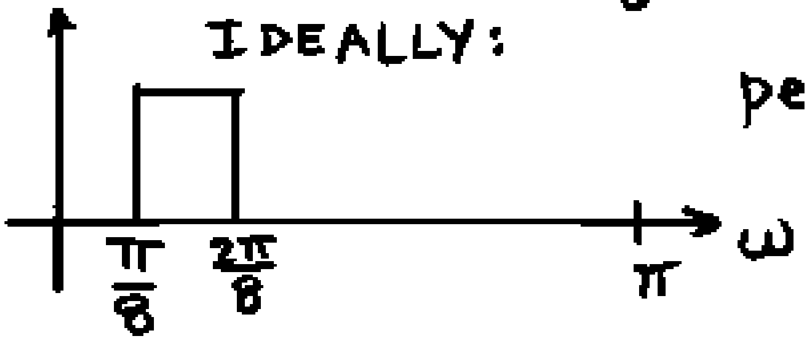
- Consider Hmwk. 7 / Final Project
- See Fig. 1 and Fig 2
- L=3 level tree-structured filter bank and equivalent 4 channel nonuniform filter bank

$$H_0(\omega) = H_0^{(2)}(\omega)\, H_0^{(2)}(2\omega)\, H_0^{(2)}(4\omega)$$



period = 2π

$$H_1(\omega) = H_0^{(2)}(\omega)\, H_0^{(2)}(2\omega)\, H_1^{(2)}(4\omega)$$



period = 2π

• More generally, for L-levels (stages)

$$H_0(\omega) = \left\{ \prod_{m'=1}^{L-1} H_0^{(2)}\left(2^{m'-1}\omega\right) \right\} H_0^{(2)}\left(2^{L-1}\omega\right)$$

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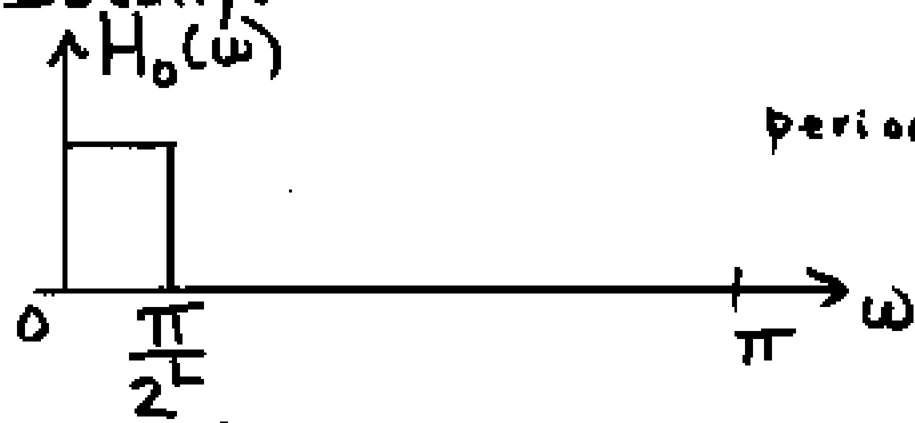
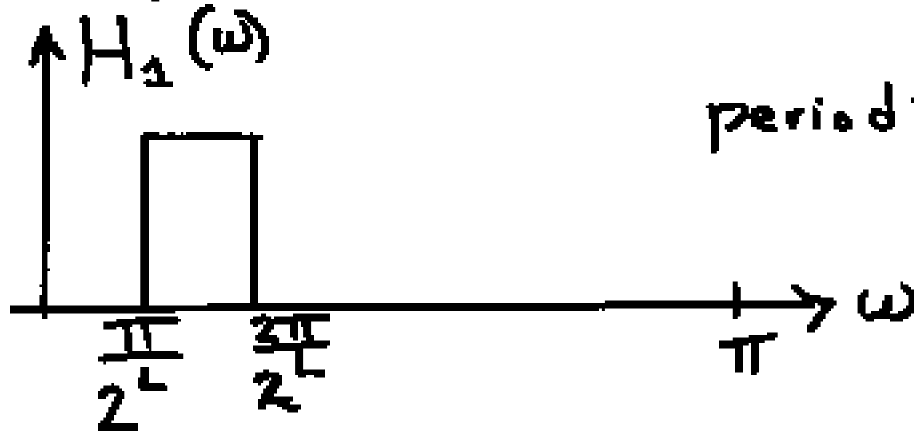
• Ideally:



$H_0(\omega)$
period = 2π
0
$\frac{\pi}{2^L}$
π
ω



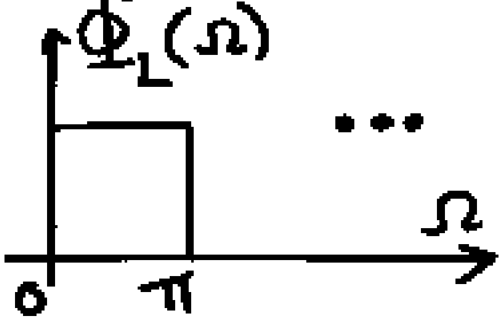
$H_1(\omega)$
period = 2π
$\frac{\pi}{2^L}$
$\frac{2\pi}{2^L}$
π
ω

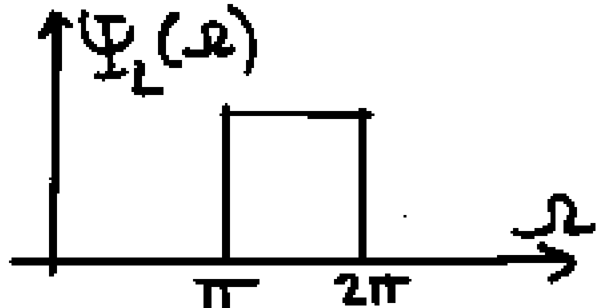
• Consider partial products:

$$\Phi_L(\Omega) = \left(\frac{1}{\sqrt{2}}\right)^L H_0\left(\frac{\omega}{2^L}\right)\Big|_{\omega=\Omega}$$

$$\Psi_L(\Omega) = \left(\frac{1}{\sqrt{2}}\right)^L H_1\left(\frac{\omega}{2^L}\right)\Big|_{\omega=\Omega}$$

$\Rightarrow$ expansion by a factor of 2^L





• still periodic : period $= 2^L(2\pi)$

$$\Phi_L(\Omega) = \left(\frac{1}{\sqrt{2}}\right)^L H_0(2^{-L}\omega)\Big|_{\omega=\Omega}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^L \left\{\prod_{m'=1}^{L-1} H_0^{(2)}(2^{m'-1}2^{-L}\omega)\right\} H_0^{(2)}(2^{L-1}2^{-L}\omega)\Big|_{\omega=\Omega}$$

$$= \left(\frac{1}{\sqrt{2}}\right) H_0^{(2)}(2^{-1}\omega) \prod_{m=2}^{L} \left(\frac{1}{\sqrt{2}}\right) H_0^{(2)}(2^{-m}\omega)\Big|_{\omega=\Omega}$$

• Similarly:

$$\Psi(\Omega) = \left(\frac{1}{\sqrt{2}}\right)^L H_1(2^{-L}\omega)\Big|_{\omega=\Omega}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^L \left\{\prod_{m'=1}^{L-1} H_0^{(2)}(2^{m'-1}2^{-L}\omega)\right\} H_1^{(2)}(2^{L-1}2^{-L}\omega)\Big|$$

$$= \left(\frac{1}{\sqrt{2}}\right) H_1^{(2)}(2^{-1}\omega) \prod_{m=2}^{L} \left(\frac{1}{\sqrt{2}}\right) H_0^{(2)}(2^{-m}\omega)\Big|_{\omega=\Omega}$$

- let $L \to \infty$ to avert periodicity
- graphically: see Fig. 11.5-1 in V. Text

- Comment on choice of filters for 2-channel PR filter bank
- here considered FIR case with

$h_1^{(2)}[n] = (-1)^n h_0^{(2)}[n]$ AND

$g_0^{(2)}[n] = h_0^{(2)}[n]$ and $g_1^{(2)}[n] = -h_1^{(2)}[n]$

- requirements $h_0^{(2)}[n]$ must satisfy:
- half-band filter $\left(\begin{matrix}\text{roughly} \\ \text{passing}\end{matrix}\ \omega \in (0, \frac{\pi}{2})\right)$
- power-symmetry

$$\left|H_0^{(2)}(\omega)\right|^2 + \left|H_0^{(2)}(\omega-\pi)\right|^2 = 1 \quad \forall \omega$$

- even length symmetric FIR

$$h_0^{(2)}[N-1-n] = h_0^{(2)}[n], \quad n = 0, 1, \ldots, N-1$$

N even!!

• note: with this choice of $h_0^{(2)}[n]$ and $h_1^{(2)}[n]$, they are orthogonal!

$$\sum_{n=0}^{N-1} h_0^{(2)}[n]\, h_1^{(2)*}[n] = 0$$

$$= \sum_{n=0}^{N-1} \underbrace{(-1)^n}_{\text{odd}} \times \underbrace{\left(h_0^{(2)}[n]\right)^2}_{\text{even}} = \text{odd}$$

hinges on both even-length and symmetry

• Example: $h_0^{(2)}[n] = \{1, 2, 2, 1\}$

$h_1^{(2)}[n] = \{1, -2, 2, -1\}$

$$\Rightarrow \sum_{n=0}^{3} h_0^{(2)}[n]\, h_1^{(2)}[n] = 0$$

- References:

- Wavelets and Subband Coding, by Martin Vetterli and Jelena Kovacevic, Prentice-Hall, 1995. ISBN 0-13-097080-8.

- Intro. to Wavelets and Wavelet Transforms, A Primer, by Sidney Burrus, Ramesh A. Gopinath, and Haitao Guo, Prentice-Hall, 1998, ISBN 0-13-489600-9.

- "Wavelet Theory: Mapping Signals to a Time-Scale Plane," by Olivier Rioul and Martin Vetterli, IEEE Signal Processing Magazine, Oct. 1991.