

EE648 (CC761-M) DSP II
Session 4 (Live: 1/21/99)

Outline:

- Development of Recursive Least Squares (RLS) Algorithm
 - Sect. 12.2.4 of 1st Ed. of P+M
- Adaptive Noise Cancellation Application Revisited

- Mathematical precursor:
Matrix Inversion Lemma

$$\begin{aligned}
 & \left(\underbrace{w \underline{R}}_{M \times M} + \underbrace{\underline{x} \underline{x}^T}_{\substack{M \times 1 \quad 1 \times M \\ M \times M}} \right)^{-1} \\
 &= \frac{1}{w} \underline{R}^{-1} - \frac{1}{w} \frac{\underline{R}^{-1} \underline{x} \underline{x}^T \underline{R}^{-1}}{w + \underbrace{\underline{x}^T \underline{R}^{-1} \underline{x}}_{1 \times M \quad M \times M \quad M \times 1}}
 \end{aligned}$$

$$\begin{aligned}
 & \left(\underline{w} \underline{R} + \underline{x} \underline{x}^T \right) \left(\frac{1}{\underline{w}} \underline{R}^{-1} \right) \\
 & \quad \frac{1}{\underline{w}} \frac{\underline{R}^{-1} \underline{x} \underline{x}^T \underline{R}^{-1}}{\underline{w} + \underline{x}^T \underline{R}^{-1} \underline{x}} \\
 = & \underline{I} + \frac{1}{\underline{w}} \underline{x} \underline{x}^T \underline{R}^{-1} \\
 & - \frac{\underline{x} \underline{x}^T \underline{R}^{-1}}{\underline{w} + \underline{x}^T \underline{R}^{-1} \underline{x}} \\
 & - \frac{1}{\underline{w}} \frac{\underline{x}^T \underline{R}^{-1} \underline{x} \underline{x} \underline{x}^T \underline{R}^{-1}}{\underline{w} + \underline{x}^T \underline{R}^{-1} \underline{x}}
 \end{aligned}$$

$$\frac{1}{w} - \frac{1}{w + \underline{x}^T \underline{R}^{-1} \underline{x}} - \frac{1}{w} \frac{\underline{x}^T \underline{R}^{-1} \underline{x}}{w + \underline{x}^T \underline{R}^{-1} \underline{x}}$$

$$\frac{w + \underline{x}^T \underline{R}^{-1} \underline{x} - w - \underline{x}^T \underline{R}^{-1} \underline{x}}{w (w + \underline{x}^T \underline{R}^{-1} \underline{x})}$$

$$= 0$$

So the Matrix Inversion Lemma holds!

- Development of RLS Algorithm
- in RLS, \underline{R}_{xx} and \underline{r}_{dx} are estimated at time n as :

$$\hat{\underline{R}}_{xx}[n] = \sum_{l=0}^n \underline{x}[l] \underline{x}^T[l] w^{n-l}$$

$$\hat{\underline{r}}_{dx}[n] = \sum_{l=0}^n d[l] \underline{x}[l] w^{n-l}$$

- where $0 < w < 1$

• $w < 1$ is used in practice to weight past data samples less than the current data samples (to adapt to time-variations in the statistics of the underlying signal)

• RLS works to minimize the time-avg'd error

$$E[n] = \sum_{\ell=0}^n w^{n-\ell} \left\{ d[\ell] - \frac{1}{M} \mathbf{G}^T \mathbf{x}[\ell] \right\}^2$$

- taking gradient of $\hat{\epsilon}[n]$ wrt $\underline{h}_M[n]$ and setting $= 0$ yields:

$$\underline{\hat{R}}_{xx}[n] \underline{h}_M[n] = \underline{\hat{r}}_d[n]$$

- observe: $\underline{\hat{R}}_{xx}[n] = \sum_{\ell=0}^n w^{n-\ell} \underline{x}[\ell] \underline{x}^T[\ell]$

$$= w \sum_{\ell=0}^{n-1} w^{n-1-\ell} \underline{x}[\ell] \underline{x}^T[\ell] + \underline{x}[n] \underline{x}^T[n]$$

$$= w \underline{R}_{xx}[n-1] + \underline{x}[n] \underline{x}^T[n]$$

• similarly:

$$\hat{\underline{r}}_{dx}[n] = w \hat{\underline{r}}_{dx}[n-1] + d[n] \underline{x}[n]$$

• $\hat{\underline{h}}_M[n] = \underline{R}_{xx}^{-1}[n] \hat{\underline{r}}_{dx}[n]$ can be computed recursively from

$$\hat{\underline{h}}_M[n-1] = \hat{\underline{R}}_{xx}^{-1}[n-1] \hat{\underline{r}}_{dx}[n-1]$$

using matrix inversion lemma

$$\hat{\underline{R}}_{xx}^{-1}[n] = \left\{ w \underline{R}_{xx}[n-1] + \underline{x}[n] \underline{x}^T[n] \right\}^{-1}$$

$$\hat{R}_{xx}^{-1}[n] = \left\{ \frac{1}{W} \hat{R}_{xx}^{-1}[n-1] - \frac{1}{W} \frac{\hat{R}_{xx}^{-1}[n-1] \underline{x}[n] \underline{x}^T[n] \hat{R}_{xx}^{-1}[n-1]}{W + \underline{x}^T[n] \hat{R}_{xx}^{-1}[n-1] \underline{x}[n]} \right\}$$

$$\underline{h}_M[n] = \underline{R}_{xx}^{-1}[n] \underbrace{\hat{\underline{r}}_{dx}[n]}_{\{W \hat{\underline{r}}_{dx}[n-1] + d[n] \underline{x}[n]\}}$$

$$= \underline{h}_M[n-1] + 3 \text{ other terms}$$

$$\bullet \text{ define: } \mu[n] = \underline{x}^T[n] \underline{R}_{xx}^{-1}[n-1] \underline{x}[n]$$

$$+ \frac{1}{w} R_{xx}^{-1}[n-1] \underline{x}[n] d[n]$$

$$- \frac{1}{w + \mu[n]} R_{xx}^{-1}[n-1] \underline{x}[n] \underline{x}^T[n] \underline{h}_M[n-1]$$

$$- \frac{1}{w} \frac{1}{w + \mu[n]} R_{xx}^{-1}[n-1] \underline{x}[n] \mu[n] d[n]$$

$$= \frac{1}{w + \mu[n]} R_{xx}^{-1}[n-1] \underline{x}[n] :$$

$$\left\{ \frac{d[n]}{w} [w + \mu[n] - \mu[n]] - \underline{x}^T[n] \underline{h}_M[n-1] \right\}$$

$$\underline{h}_M[n] = \underline{h}_M[n-1] + \left\{ \frac{1}{w + \mu[n]} \right\} \hat{R}_{xx}^{-1}[n-1] \underline{x}[n] \left\{ d[n] - \underline{h}_M^T[n-1] \underline{x}[n] \right\}$$

where: $\mu[n] = \underline{x}^T[n] \underline{R}_{xx}[n-1] \underline{x}[n]$

define: $e[n, n-1] = d[n] - \underline{h}_M^T[n-1] \underline{x}[n]$

$$\left. \begin{aligned} \underline{h}_M[n] = \underline{h}_M[n-1] + \\ \frac{e[n, n-1]}{w + \mu[n]} \hat{R}_{xx}^{-1}[n-1] \underline{x}[n] \end{aligned} \right\} \begin{array}{l} \text{RLS} \\ \text{update} \end{array}$$

Summary of RLS

0. Initialization: $\underline{h}_M[-1] = (\underline{0}_M, \text{e.g.})$

$$\text{and } \hat{R}^{-1}[-1] = \frac{1}{\sigma} \underline{I}_M$$

1. $e[n, n-1] = d[n] - \underline{h}_M^T[n-1] \underline{x}[n]$

2. a. $\underline{f}[n] = \hat{R}_{xx}^{-1}[n-1] \underline{x}[n]$

b. $\mu[n] = \underline{x}^T[n] \underline{f}[n]$

c. $\underline{K}_M[n] = \underline{f}[n] / (w + \mu[n])$

3. $\underline{h}_M[n] = \underline{h}_M[n-1] + e[n, n-1] \underline{K}_M[n]$

4. $\underline{R}_{xx}^{-1}[n] = \frac{1}{w} \{ \underline{R}_{xx}^{-1}[n-1] + \underline{K}_M[n] \underline{f}^T[n] \}$

Go to 1.