EE648 (CC761-M) DSP II

Session 4 (Live: 1/21/99)

Outline:

- Development of Recursive Least Squares (RLS) Algorithm
  - Sect. 12.2.4 of 1st Ed. of P+M
- Adaptive Noise Cancellation Application Revisited

- Mathematical precursor: Matrix Inversion Lemma

$$\Big(\underbrace{w\underline{R}}_{M\times M} + \underbrace{\underline{x}\,\underline{x}^T}_{\substack{M\times 1 \;\; 1\times M \\ M\times M}}\Big)^{-1}$$

$$= \frac{1}{w}\underline{R}^{-1} - \frac{1}{w}\frac{\underline{R}^{-1}\underline{x}\,\underline{x}^T\underline{R}^{-1}}{w + \underbrace{\underline{x}^T}_{1\times M}\underbrace{\underline{R}^{-1}}_{M\times M}\underbrace{\underline{x}}_{M\times 1}}$$

$$(w\underline{R} + \underline{x}\,\underline{x}^T)\left(\frac{1}{w}\underline{R}^{-1} - \frac{1}{w}\frac{\underline{R}^{-1}\underline{x}\,\underline{x}^T\underline{R}^{-1}}{w+\underline{x}^T\underline{R}^{-1}\underline{x}}\right)$$

$$= \underline{I} + \frac{1}{w}\underline{x}\,\underline{x}^T R^{-1} - \frac{\underline{x}\,\underline{x}^T R^{-1}}{w+\underline{x}^T\underline{R}^{-1}\underline{x}} - \frac{1}{w}\frac{\underline{x}^T\underline{R}^{-1}\underline{x}\;\underline{x}\,\underline{x}^T R^{-1}}{w+\underline{x}^T\underline{R}^{-1}\underline{x}}$$

$$\frac{1}{w} - \frac{1}{w+\underline{x}^T\underline{R}^{-1}\underline{x}} - \frac{1}{w}\frac{\underline{x}^T\underline{R}^{-1}\underline{x}}{w+\underline{x}^T\underline{R}^{-1}\underline{x}}$$

$$\frac{w+\underline{x}^T\underline{R}^{-1}\underline{x} - w - \underline{x}^T\underline{R}^{-1}\underline{x}}{w(w+\underline{x}^T\underline{R}^{-1}\underline{x})}$$

$$= 0$$

So the Matrix Inversion Lemma holds!

- Development of RLS Algorithm
- in RLS, $\underline{R}_{xx}$ and $\underline{r}_{dx}$ are estimated at time n as :

$$\hat{\underline{R}}_{xx}[n] = \sum_{\ell=0}^{n} \underline{x}[\ell]\,\underline{x}^T[\ell]\,w^{n-\ell}$$

$$\hat{\underline{r}}_{dx}[n] = \sum_{\ell=0}^{n} d[\ell]\,\underline{x}[\ell]\,w^{n-\ell}$$

- where $0 < w < 1$

- $w < 1$ is used in practice to weight past data samples less than the current data samples (to adapt to time-variations in the statistics of the underlying signal)
- RLS works to minimize the time-avg'd. error

$$\mathcal{E}[n] = \sum_{\ell=0}^{n} w^{n-\ell}\left\{d[\ell] - \underline{h}_M^T[\ell]\,\underline{x}[\ell]\right\}^2$$

- taking gradient of $\mathcal{E}[n]$ wrt $\underline{h}_M[n]$ and setting $=0$ yields:

$$\hat{\underline{R}}_{xx}[n]\,\underline{h}_M[n] = \hat{\underline{r}}_{dx}[n]$$

- observe: $\hat{\underline{R}}_{xx}[n] = \sum_{\ell=0}^{n} w^{n-\ell}\,\underline{x}[\ell]\,\underline{x}^T[\ell]$

$$= w\sum_{\ell=0}^{n-1} w^{n-1-\ell}\,\underline{x}[\ell]\underline{x}^T[\ell] + \underline{x}[n]\underline{x}^T[n]$$

$$= w\,\underline{R}_{xx}[n-1] + \underline{x}[n]\,\underline{x}^T[n]$$

- similarly:

$$\hat{\underline{r}}_{dx}[n] = w\,\hat{\underline{r}}_{dx}[n-1] + d[n]\,\underline{x}[n]$$

- $\underline{h}_M[n] = \underline{R}_{xx}^{-1}[n]\,\hat{\underline{r}}_{dx}[n]$ can be computed resursively from

$$\underline{h}_M[n-1] = \hat{\underline{R}}_{xx}^{-1}[n-1]\,\hat{\underline{r}}_{dx}[n-1]$$

using matrix inversion lemma

$$\hat{\underline{R}}_{xx}^{-1}[n] = \left\{w\,\underline{R}_{xx}[n-1] + \underline{x}[n]\,\underline{x}^T[n]\right\}^{-1}$$

$$\hat{\underline{R}}_{xx}^{-1}[n] = \underbrace{\left\{\frac{1}{w}\hat{\underline{R}}_{xx}^{-1}[n-1] - \frac{1}{w}\frac{\hat{\underline{R}}_{xx}^{-1}[n-1]\underline{x}[n]\underline{x}^T[n]\hat{\underline{R}}_{xx}^{-1}[n-1]}{w + \underline{x}^T[n]\hat{\underline{R}}_{xx}^{-1}[n-1]\underline{x}[n]}\right\}}$$

$$\underline{h}_M[n] = \underline{R}_{xx}^{-1}[n]\ \underbrace{\hat{\underline{r}}_{dx}[n]}_{\{w\,\hat{\underline{r}}_{dx}[n-1] + d[n]\,\underline{x}[n]\}}$$

$$= \underline{h}_M[n-1] + \text{ 3 other terms}$$

• define: $\mu[n] = \underline{x}^T[n]\ \underline{R}_{xx}^{-1}[n-1]\underline{x}[n]$

$$+ \frac{1}{w}\ \underline{R}_{xx}^{-1}[n-1]\ \underline{x}[n]\ \ d[n]$$

$$- \frac{1}{w+\mu[n]}\ \underline{R}_{xx}^{-1}[n-1]\underline{x}[n]\ \ \underline{x}^T[n]\,\underline{h}_M[n-1]$$

$$- \frac{1}{w}\ \frac{1}{w+\mu[n]}\ \underline{R}_{xx}^{-1}[n-1]\,\underline{x}[n]\ \mu[n]\ d[n]$$

$$= \frac{1}{w+\mu[n]}\ \underline{R}_{xx}^{-1}[n-1]\underline{x}[n]\ \cdot \left\{\frac{d[n]}{w}\left[w+\mu[n]-\mu[n]\right] - \underline{x}^T[n]\,\underline{h}_M[n-1]\right\}$$

$$\underline{h}_M[n] = h_M[n-1] + \left\{\frac{1}{w+\mu[n]}\right\}\hat{\underline{R}}_{xx}^{-1}[n-1]\underline{x}[n]\left\{d[n] - \underline{h}_M^T[n-1]\underline{x}[n]\right\}$$

• where: $\mu[n] = \underline{x}^T[n]\ \underline{R}_{xx}^{-1}[n-1]\ \underline{x}[n]$

• define: $e[n,n-1] = d[n] - \underline{h}_M^T[n-1]\ \underline{x}[n]$

$$\left.\begin{aligned} h_M[n] &= \underline{h}_M[n-1] + \\ &\frac{e[n,n-1]}{w+\mu[n]}\ \hat{\underline{R}}_{xx}^{-1}[n-1]\underline{x}[n] \end{aligned}\right\} \text{RLS update}$$

Summary of RLS

0. Initialization: $\underline{h}_M[-1] = (\underline{0}_M, \text{e.g.})$
   and $\hat{\underline{R}}^{-1}[-1] = \frac{1}{\sigma}\ \underline{I}_M$
1. $e[n,n-1] = d[n] - \underline{h}_M^T[n-1]\ \underline{x}[n]$
2. a. $\underline{f}[n] = \hat{\underline{R}}_{xx}^{-1}[n-1]\ \underline{x}[n]$
   b. $\mu[n] = \underline{x}^T[n]\ \underline{f}[n]$
   c. $\underline{K}_M[n] = \underline{f}[n] / (w+\mu[n])$
3. $\underline{h}_M[n] = \underline{h}_M[n-1] + e[n,n-1]\ \underline{K}_M[n]$
4. $\underline{R}_{xx}^{-1}[n] = \frac{1}{w}\left\{\underline{R}_{xx}^{-1}[n-1] + \underline{K}_M[n]\ \underline{f}^T[n]\right\}$

Go to 1.