1. **Text, problem 2.97 (page 91)**

   (a) We wish to find the probability that the block is accepted, or equivalently, the probability that there are 1 or fewer errors. Note that there are 100 ways to receieve a single error (depending on which of the 100 bits were incorrect), but only one way to receive no errors, so that

   $$\begin{aligned} P[\text{block accepted}] &= P[\text{no errors}] + P[1 \text{ error}] \\ &= (1-p)^{100} + 100(1-p)^{99}p \\ &= (0.99)^{100} + 100(0.99)^{99}(0.01) \\ &= 0.7358. \end{aligned}$$

   (b) Let the probability of a block transmission failing be q. From part (a), we have that

   $$q = P[\text{transmission fail}] = 1 - P[\text{block accepted}] = 0.2642.$$

   If a block transmission fails, it is retransmitted, so that the number, M, of retransmissions will consist of $M-1$ failed transmission followed by 1 successful one. The probability of M transmissions is then $q^{M-1}(1-q)$

2. **Text, problem 3.18 (page 132)** As in the hint, define I to be the index of the first refresh message which is not lost. In order to extend the reservation, we must send j messages before the 10 minute reservation expires, and have that $I \leq j$ to guarantee an extension of reservation. Therefore, the probability that the reservation is extended within j sent messages is the probability

   $$P[I \leq j] = P[I=1] + P[I=2] + \ldots + P[I=j] = \sum_{i=1}^{j} p_I[i],$$

   where $p_I[i]$ is the pmf of random variable I.

   We now need only find the form for the pmf $p_I[i] = P[I=i]$. We have that $I = i$ if and only if $i-1$ refresh requests fail, followed by one that succeeds. If we let $P[\text{request fails}] = p = 1/2$, then

   $$p_I(i) = P[I=i] = p^{i-1}(1-p) = (1/2)^i.$$

Therefore

$$P[I \leq j] = \sum_{i=1}^{j} p_I[i] = \sum_{i=1}^{j} (1/2)^i = \frac{1/2 - (1/2)^{j+1}}{1 - (1/2)} = 1 - (1/2)^j.$$

The problem asks to find j such that the probability of extension is greater than 0.99, or, in other words, find j such that

$$\begin{aligned} P[I \leq j] &> 0.99 \\ \Rightarrow 1 - (1/2)^j &> 0.99 \\ \Rightarrow (1/2)^j &< 0.01 \\ \Rightarrow j \ln(1/2) &< \ln(0.01) \\ \Rightarrow j &> \frac{\ln(0.01)}{\ln(1/2)} \\ \Rightarrow j &> 6.6439. \end{aligned}$$

Since we must get at least $j = 7$ requests in to guarantee 0.99 probability of reservation, we should start sending refresh requests at least 70 seconds prior to expiration.

3. (a) Mixed Random Variable

Since there are jumps and continuous parts of the cdf.

(b) $P[X < -1] = \cancel{F_X(-1)} = \boxed{0}$

$P[X=-1] = \underset{\varepsilon^+ \to 0}{F_X(-1+\varepsilon)} - \underset{\varepsilon^+ \to 0}{F_X(-1-\varepsilon)} = \frac{2}{10}$

$P[X \le -1] = F_X(-1) = \boxed{\frac{2}{10}}$

$P[-1 < X < -0.75] = P[-1 < X \le -0.75] = F_X(-0.75) - F_X(-1) = \boxed{0}$

$P[-0.5 \le X < 0] = \cancel{P[-0.5]+} P[-0.5 < X \le 0] - P[X=0] = \overset{P[-0.5)+}{} F_X(0) - F_X(-0.5) - P[X=0]$

$= \frac{6}{10} - \frac{2}{10} - \frac{2}{10} = \boxed{\frac{2}{10}}$

$P[-0.5 \le X \le 0.5] = P[X=-0.5] + P[-0.5 < X \le 0.5] = 0 + F_X(0.5) - F_X(-0.5)$

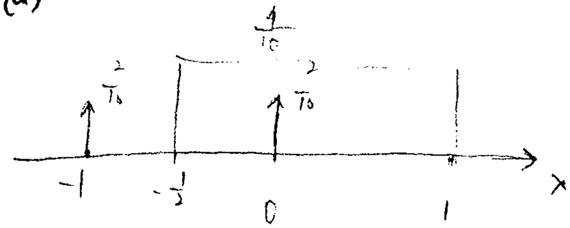
$= \frac{8}{10} - \frac{2}{10} = \boxed{\frac{6}{10}}$

$P[|X-0.5| < 0.5] = P[0 < X < 1] = P[0 < X \le 1] = F_X[1] - F_X[0]$

$= 1 - \frac{6}{10} = \boxed{\frac{4}{10}}$

4. $f_X(x)$ (pdf)

(a)



(b) $P(X<-1) = P(X\le -1) - P(X=-1) = F_X(-1) - P(X=-1) = \int_{-\infty}^{-1^+} f_X(x)\,dx - \frac{2}{10} = 0$

$P(X\le -1) = F_X(-1) = \int_{-\infty}^{-1^+} f_X(x)\,dx = \boxed{\frac{2}{10}} \left(= \int_{-\infty}^{-1^+} \frac{2}{10}\delta(x+1)\,dx\right)$

$P(-0.5 \le X < 0) = P(X=-0.5) + P(-0.5 < X \le 0) - P(X=0)$

$= 0 + \int_{-0.5}^{0^+} f_X(x)\,dx - \frac{2}{10}$

means the area below the curve + the integral of the impulse at $x=0$

$= 0 + \frac{1}{2}\times\frac{4}{10} + \frac{2}{10} - \frac{2}{10} = \boxed{\frac{2}{10}}$

$P(-1 < X < -0.75) = P(-1 < X \le -0.75) - P(X=-0.75)$

$= \int_{-1^+}^{-0.75^+} f_X(x)\,dx - 0 = 0 - 0 = \boxed{0}$

$P[-0.5 \le X \le 0.5) = P(X=-0.5) + P(-0.5 < X \le 0.5)$

~~$= P(X=-0.5) + P$~~

$= 0 + \int_{-0.5^+}^{0.5^+} f_X(x)\,dx = \frac{4}{10}\times 1 + \frac{2}{10} = \frac{6}{10}$

area below the curve

the integral of the impulse at $x=0$

$P[\,|X-0.5| < 0.5) = P(0 < X < 1) = P(0 < X \le 1) - P(X=1)$

$= \int_{0^+}^{1^+} f_X(x)\,dx - 0 = \frac{4}{10}\times 1 - 0 = \frac{4}{10}$

$P[-0.5 \le X < 0) = \int_{-0.5^-}^{0^-} f_X(x)\,dx = \frac{4}{10}\times\frac{1}{2} = \frac{2}{10}$

5. (a) $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

$\int_0^5 k e^{-x}\,dx = 1$

$\Rightarrow k = \frac{1}{1-e^{-5}}$

(b) $P(1 \le X < 3) = \int_1^3 f_X(x)\,dx$

$= \int_1^3 \frac{1}{1-e^{-5}} e^{-x}\,dx$

$= \frac{e^{-1}-e^{-3}}{1-e^{-5}}$

$P(1 < X \le 5) = \int_1^5 f_X(x)\,dx$

$= \int_1^5 \frac{1}{1-e^{-5}} e^{-x}\,dx$

$= \frac{e^{-1}-e^{-5}}{1-e^{-5}}$

(c) $X^3 < 5 \Rightarrow X < \sqrt[3]{5}$

$P_r(X^3 < 5) = P_r(X < \sqrt[3]{5}) = \int_0^{\sqrt[3]{5}} f_X(x)\,dx$

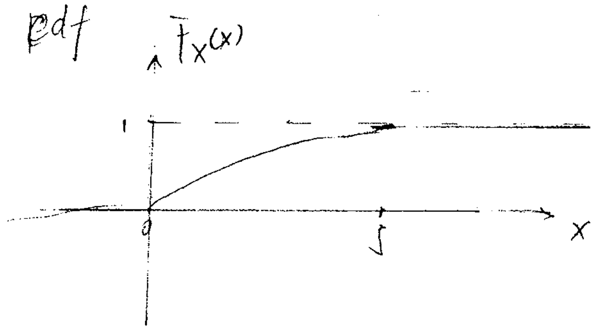
$= \int_0^{\sqrt[3]{5}} \frac{e^{-x}}{1-e^{-5}}\,dx$

$= \frac{1-e^{-\sqrt[3]{5}}}{1-e^{-5}}$

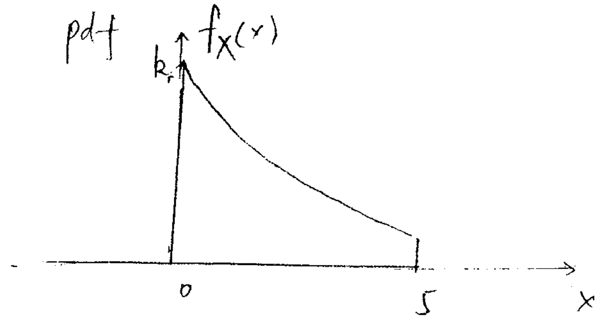
(d) $F_X(x) = \int_{-\infty}^{x} \frac{e^{-x'}}{1-e^{-5}}\,dx'$

$$= \begin{cases} 0 & x \le 0 \\ \frac{1-e^{-x}}{1-e^{-5}} & 0 < x \le 5 \\ 1 & x > 5 \end{cases}$$

cdf



pdf



6. (a) $\sum_{x_i=0}^{5} P_X(x_i) = k\left(\frac{1}{2}\right)^0 + k\left(\frac{1}{2}\right)^1 + k\left(\frac{1}{2}\right)^2 + k\left(\frac{1}{2}\right)^3 + k\left(\frac{1}{2}\right)^4 + k\left(\frac{1}{2}\right)^5 = 1$

$\Rightarrow k = \frac{32}{63}$

(b) $P(1 \le X < 3) = P_X(1) + P_X(2) = \frac{32}{63}\left(\frac{1}{2}\right)^1 + \frac{32}{63}\left(\frac{1}{2}\right)^2 = \frac{8}{21}$

$P(1 < X \le 5) = P_X(2) + P_X(3) + P_X(4) + P_X(5) = \frac{32}{63}\left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5\right)$

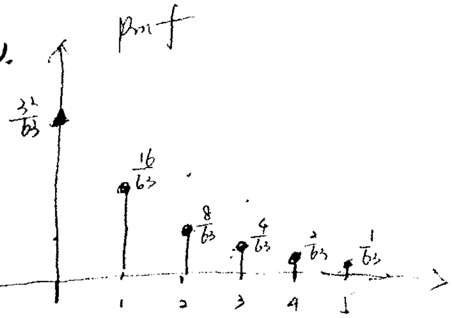
$= \frac{5}{21}$

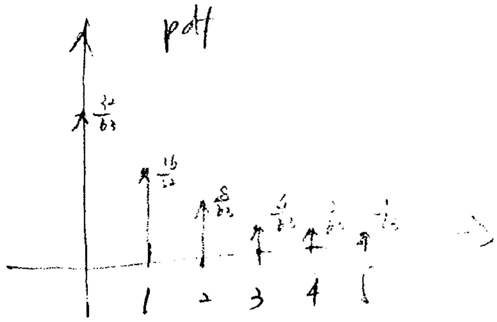
(c). $x^3 < 5$

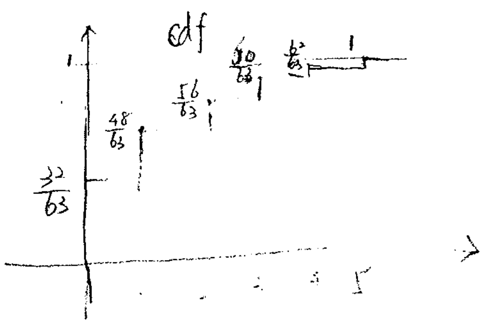
$x < \sqrt[3]{5}$

$P(x < \sqrt[3]{5}) = P(0) + P(1) = \frac{32}{63}\left(\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1\right) = \frac{16}{21}$

(d).







7. (a) $\boxed{\text{True}}$

$$P(A\cap B|C) = \frac{P(A\cap B\cap C)}{P(C)} = \frac{P(A)P(B)P(C)}{P(C)} = P(A)P(B)$$

$$P(A|C) = \frac{P(A\cap C)}{P(C)} = P(A)$$

$$P(B|C) = \frac{P(B\cap C)}{P(C)} = P(B)$$

$$\Rightarrow P(A\cap B|C) = P(A|C)P(B|C)$$

(b) $\boxed{\text{True}}$

$$P(S\cap\phi) = P(\phi) = 0 = P(S)P(\phi) = 0$$

$$P(S\cap A) = P(A) = P(S)P(A) = 1\cdot P(A) = P(A)$$

$$P(\phi\cap A) = P(\phi) = 0 = P(\phi)P(A) = 0$$

$P(A \cap S \cap \phi) = P(\phi) = P(A)\,P(S)\,P(\phi) = 0$

$\Rightarrow$ They are all independent.

(c) $\boxed{\text{False}}$ Actually as to the (b)

S and ϕ are mutually exclusive, but they are independent

(d) $\boxed{\text{False}}$

According to (b)

S and ϕ are independent. and they are mutually exclusive.

(e) $\boxed{\text{True}}$

$Pr(A_n)\,Pr(A_{n-1}|A_n)\,Pr(A_{n-2}|A_n \cap A_{n-1}) \cdots Pr(A_1|A_n \cap \cdots \cap A_2)$

$= Pr(A_n \cap A_{n-1})\,Pr(A_{n-2}|A_n \cap A_{n-1}) \cdots Pr(A_1|A_n \cap \cdots \cap A_2) = Pr(A_n \cap A_{n-1} \cap A_{n-2}) \cdots Pr(A_1|A_n \cap A_{n-1} \cap \cdots \cap A_2)$

$= Pr(A_n \cap A_{n-1} \cap \cdots \cap A_1)$

(f) $\sum_{k=0}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} = (1-(1-p))^k = 1$

$\boxed{\text{False}}$

$\sum_{k=1}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} = 1 - \binom{n}{0} p^0 (1-p)^n = 1-(1-p)^n \neq 1$ (when $p \neq 1$)

(g) $\boxed{\text{False}}$

$f_X(x_0) = \left.\frac{d F_X(x)}{dx}\right|_{X=x_0} = \left.\frac{d P_X(X \leq x)}{dx}\right|_{X=x_0} \neq P_X(X=x_0) = 0$

(when X is Continuous R.V.)

(h) $\boxed{\text{True}}$

$Pr(a \leq X \leq b) = Pr(a \leq X < b) = Pr(a < X < b) = Pr(a < X \leq b) = \int_a^b f_X(x)\,dx$

(i) True

It's definition

(j) True

$$Pr(a \le X \le b) = \sum_{x_i=a}^{b} Pr(X=x_i) = \sum_{x_i:\, a \le x_i \le b} f_X(x_i)$$

(k) False

$$Pr(a < X \le b) = F_X(b) - F_X(a)$$

$$Pr(a \le X \le b) = Pr(a) + F_X(b) - F_X(a)$$

If X is ~~not~~ discrete r.v. at a, $Pr(a) \neq 0$

Henses, $Pr(a \le X \le b)$ might not equal $F_X(b) - F_X(a)$