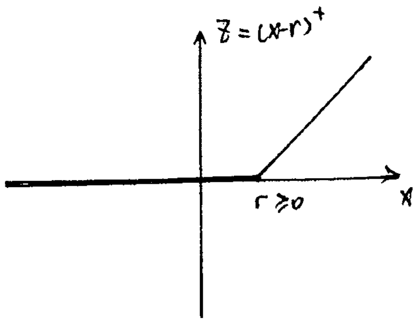
ECE 302 HW #13 Solution.

1. Problem 4.77

$$f_X(x) = \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}}, \quad x > 0, \quad \alpha > 0$$

(a) Case 1 : $r \geq 0$



$$\left|\frac{dx}{dz}\right| = 1$$

$$f_Z(z) = f_1(z) + f_2(z)$$

(i) For $z > 0$

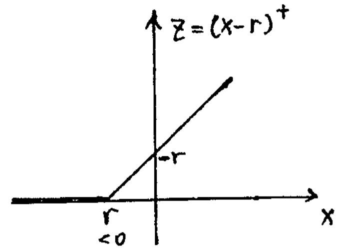
$$f_1(z) = f_X(x)\left|\frac{dx}{dz}\right|$$
$$= f_X(z+r) = \frac{z+r}{\alpha^2} e^{-\frac{(z+r)^2}{2\alpha^2}}$$

(ii) For $z = 0$

$$f_2(z) = P_r(z=0)\,\delta(z)$$
$$= P_r(0 \leq X \leq r)\,\delta(z)$$
$$= \left(\int_0^r f_X(x)\,dx\right)\delta(z)$$
$$= -e^{-\frac{x^2}{2\alpha^2}}\Big|_0^r\,\delta(z)$$
$$= \left(1 - e^{-\frac{r^2}{2\alpha^2}}\right)\delta(z)$$

$$\therefore f_Z(z) = f_1(z) + f_2(z)$$
$$= \begin{cases} \frac{z+r}{\alpha^2} e^{-\frac{(z+r)^2}{2\alpha^2}}, & z > 0 \\ \left(1 - e^{-\frac{r^2}{2\alpha^2}}\right)\delta(z), & z = 0 \\ 0, & \text{else} \end{cases}$$

Case 2 : $r < 0$



$$\left|\frac{dx}{dz}\right| = 1$$

$$f_Z(z) = f_1(z) + f_2(z)$$

(i) For $z > -r$

$$f_1(z) = f_X(x)\left|\frac{dx}{dz}\right|$$
$$= f_X(z+r) = \frac{z+r}{\alpha^2} e^{-\frac{(z+r)^2}{2\alpha^2}}$$

(ii) For $0 < z \leq -r$

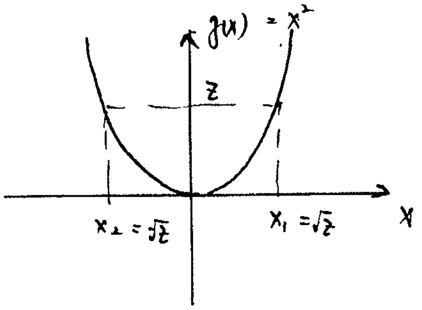
$$f_1(z) = f_X(x)\left|\frac{dx}{dz}\right|$$
$$= 0 \cdot 1 = 0$$

(iii) For $z = 0$

$$f_2(z) = P_r(z=0)\,\delta(z)$$
$$= P_r(X < r)\,\delta(z)$$
$$= 0 \cdot \delta(z) = 0$$

$$\therefore f_Z(z) = f_1(z) + f_2(z)$$
$$= \begin{cases} \frac{(z+r)^2}{\alpha^2} e^{-\frac{(z+r)^2}{2\alpha^2}}, & z > -r \\ 0, & \text{else} \end{cases}$$

(b)



$z = x^2$

$\left|\frac{dz}{dx}\right| = |2x| = 2\sqrt{z} \quad \left|\frac{dx}{dz}\right| = \frac{1}{2\sqrt{z}}$

$z = x^2$

$x_1 = \sqrt{z}, \quad x_2 = \sqrt{z}$

(i) for $z \geq 0$

$f(z) = (f_x(x_1) + f_x(x_2)) \left|\frac{dx}{dz}\right|$

$= (f_x(\sqrt{z}) + f_x(-\sqrt{z})) \left|\frac{dx}{dz}\right|$, where $f_x(-\sqrt{z}) = 0$, for $f_x(x) = 0$, $(x<0)$

$= \frac{\sqrt{z}}{\alpha^2} e^{-\frac{z}{2\alpha^2}} \cdot \frac{1}{2\sqrt{z}}$

$= \frac{1}{2\alpha^2} e^{-\frac{z}{2\alpha^2}}$

(ii) for $z \leq 0$

$f(z) = 0$

$\therefore f_z(z) = \begin{cases} \frac{1}{2\alpha^2} e^{-\frac{z^2}{2\alpha^2}} & , z > 0 \\ 0 & , \text{else} \end{cases}$

ECE302 HW3.

2. (a) ① $f_X(x) = \begin{cases} 2e^{-2x} & 0<x<\infty \\ 0 & \text{else} \end{cases}$

$Y = X^3$

$\Rightarrow X = \sqrt[3]{Y}$

$f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|$

$= 2e^{-2x} \frac{1}{3} y^{-\frac{2}{3}}$

$= 2e^{-2y^{\frac{1}{3}}} \frac{1}{3} y^{-\frac{2}{3}}$

$= \frac{2}{3} y^{-\frac{2}{3}} e^{-2y^{\frac{1}{3}}}$ $\Rightarrow f_Y(y) = \begin{cases} \frac{2}{3} y^{-\frac{2}{3}} e^{-2y^{\frac{1}{3}}} & y>0 \\ 0 & y \le 0 \end{cases}$

Alternatively,

$F_X(x) = \int_{-\infty}^{x} f_X(x')\,dx'$

$= \int_0^x 2e^{-2x'}\,dx'$

$= -e^{-2x'}\Big|_0^x$

$= 1 - e^{-2x}$

$F_Y(y) = P(Y \le y)$ $= \int_{-\infty}^{\sqrt[3]{y}} f_X(x)\,dx$

$= P(X^3 \le y)$ $= \int_0^{\sqrt[3]{y}} 2e^{-2x}\,dx$

$= P(X \le \sqrt[3]{y})$ $= 1 - e^{-2\sqrt[3]{y}}$ $y>0$

$= F_X(\sqrt[3]{y})$ $(= 0$ $y \le 0)$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -e^{-2\sqrt[3]{y}} \cdot (-\tfrac{2}{3}) y^{-\frac{2}{3}} = \frac{2}{3} y^{-\frac{2}{3}} e^{-2\sqrt[3]{y}}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{2}{3} y^{-\frac{2}{3}} e^{-2\sqrt[3]{y}} & y > 0 \\ 0 & y \le 0 \end{cases}$$

② the probability that $Y > 2X$

$Y = X^3 > 2X$

$\Rightarrow X > \sqrt{2}$ (for $X > 0$)

$P(Y > 2X)$

$= P_X(X > \sqrt{2})$

$= 1 - F_X(\sqrt{2})$

$= 1 - \int_{-\infty}^{\sqrt{2}} f_X(x)\,dx$

$= 1 - \int_0^{\sqrt{2}} 2e^{-2x}\,dX$

$= 1 + \int_0^{\sqrt{2}} e^{-2X}\,d(-2X)$

$= 1 + e^{-2X}\Big|_0^{\sqrt{2}}$

$= e^{-2\sqrt{2}}$

(b) ① $Y = X^3$

$X = \sqrt[3]{Y}$

Since $P_X(x_i) = (\frac{1}{2})^{x_i+1}$

$\Rightarrow P_Y(y) = \begin{cases} (\frac{1}{2})^{\sqrt[3]{y}+1} & y = 0, 1, 8, 27, \cdots \\ 0 & \text{else}. \end{cases}$

② $Y = X^3 > 2X$

$X^3 - 2X > 0$

$X > \sqrt{2}$ (since $X \geq 0$)

$P(X > \sqrt{2}) = P_X(2) + P_X(3) + \cdots$, or $P(X>\sqrt{2}) = 1 - P(X=1) - P(X=0)$

$= (\frac{1}{2})^3 + (\frac{1}{2})^4 + \cdots$ $= 1 - (\frac{1}{2})^{1+1} - (\frac{1}{2})^1$

$= \dfrac{(\frac{1}{2})^3}{1-\frac{1}{2}}$ $= \frac{1}{4}$

$= \frac{1}{4}$

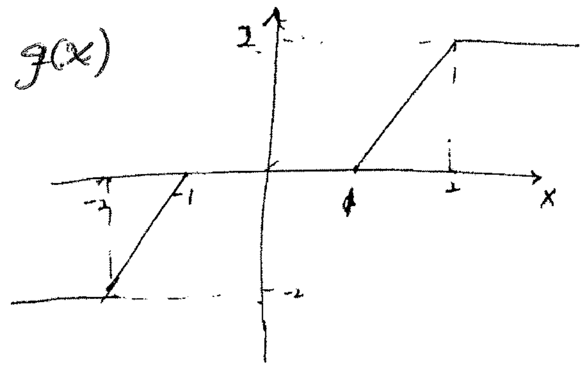
2. (c) We now let

$$Y = |X|^3 = \begin{cases} X^3, & X \geq 0 \\ -(X^3), & X < 0 \end{cases}$$

Since $P[X < 0] = 0$ in both cases for the distribution of X in parts (a) and (b), then we find that $Y = |X|^3 = X^3$ *almost always* (i.e. for all X with nonzero probability), and so the answers will remain the same when $Y = |X|^3$ as when $Y = X^3$ above.

5

3. We have $Y = g(X)$, with $g(x)$ shown below:



a. Y is thus a mixed random variable since $P_r[Y=2]$, $P_r[Y=-2]$ and $P_r[Y=0]$ are all nonzero, but $P_r[Y=y]=0$ for $y \in (-2,-1)$ and $y \in (1,2)$.

b. Consider 6 cases for $f_Y(y)$:

① $y > 2$

② $y = 2$

③ $y \in (0,2)$

④ $y = 0$

⑤ $y \in (-2, 0)$

⑥ $y = -2$

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① $y > 2$ $\qquad \phi \quad f_Y(y) = 0$

② $y = 2$ $\qquad f_Y(y) = P_r(Y=2)\,\delta(y-2)$

$= P_r(X \geq 2)\,\delta(y-2)$

$= \int_2^{+\infty} \frac{1}{2} e^{-x}\,dx\; \delta(y-2)$

$= \frac{1}{2} e^{-2}\,\delta(y-2)$

③ $0 < y < 2$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$y = 2x - 2$

$x = \frac{1}{2}y + 1$

$$= \frac{1}{2}e^{-|x|}\left|\frac{d(\frac{1}{2}y+1)}{dy}\right|$$

$$= \frac{1}{2}e^{-\frac{1}{2}y-1}\ \frac{1}{2}$$

$$= \frac{1}{4}e^{-\frac{1}{2}y-1}$$

④ $y = 0$

$$f_Y(y) = P_r(y = 0)\ \delta(y-0)$$

$$= P_r(-1 \le x \le 1)\ \delta(y-0)$$

$$= \int_{-1}^{1} f_X(x)\,dx\ \delta(y-0)$$

$$= \left(\int_{-1}^{0} \frac{1}{2}e^{x}dx + \int_{0}^{1}\frac{1}{2}e^{-x}dx\right)\ \delta(y-0)$$

$$= \left(\frac{1}{2} - \frac{1}{2}e^{-1} - \frac{1}{2}e^{-1} + \frac{1}{2}\right)\ \delta(y-0)$$

$$= (1 - e^{-1})\ \delta(y-0)$$

⑤ $-2 < y < 0$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$y = 2x + 2$

$x = \frac{1}{2}y - 1 \quad < 0$

$$= \frac{1}{2}e^{\frac{1}{2}y-1}\ \frac{1}{2}$$

$$= \frac{1}{4}e^{\frac{1}{2}y-1}$$

⑥ $y = -2$

$$f_Y(y) = P_r(y = -2)\ \delta(y+2)$$

$$= P_r(x \le -2)\ \delta(y+2)$$

$$= \int_{-\infty}^{-2} f_X(x)\,dx\ \delta(y+2)$$

$$= \int_{-\infty}^{-2} \frac{1}{2}e^{x}dx\ \delta(y+2$$

$$= \frac{1}{2}e^{-2}\ \delta(y+2)$$

1

⑦ $y < -2$ ϕ

$f_Y(y) = 0$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{2} e^{-2} \delta(y-2) & , \quad y=2 \\ \frac{1}{4} e^{-\frac{1}{2}y-1} & , \quad 0<y<2 \\ (1-e^{-1}) \delta(y-0) & , \quad y=0 \\ \frac{1}{4} e^{\frac{1}{2}y-1} & , \quad -2<y<0 \\ \frac{1}{2} e^{-2} \delta(y+2) & , \quad y=-2 \\ 0 & , \quad \text{else} \end{cases}$$

6.

$$\{x : g(x) \leq y\} = \begin{cases} [-\infty, +\infty) & y \geq 2 \\ (-\infty, \frac{1}{2}y+1] & 0<y<2 \\ [-\infty, 1] & y=0 \\ (-\infty, \frac{1}{2}y-1] & -2<y<0 \\ (-\infty, -2] & y=-2 \\ \phi & \text{else} \end{cases}$$

$$F_Y(y) = \int_{g(x)\leq y} f_X(x)dx = \begin{cases} \int_{-\infty}^{\infty} f_X(x)dx = 1 & \underline{y \geq 2} \\ \int_{-\infty}^{\frac{1}{2}y+1} f_X(x)\,dx = \int_{-\infty}^{0} \frac{1}{2}e^{x}dx + \int_{0}^{\frac{1}{2}y+1} \frac{1}{2}e^{-x}dx = 1-\frac{1}{2}e^{-(\frac{1}{2}y+1)} & \underline{0<y<2} \\ \int_{-\infty}^{0} \frac{1}{2}e^{x}dx + \int_{0}^{1}\frac{1}{2}e^{-x}dx = 1-\frac{1}{2}e^{-1} & \underline{y=0} \\ \int_{-\infty}^{\frac{1}{2}y-1} \frac{1}{2}e^{x}dx = \frac{1}{2}e^{\frac{1}{2}y-1} & \underline{-2<y<0} \\ \int_{-\infty}^{-2} \frac{1}{2}e^{x}dx = \frac{1}{2}e^{-2} & \underline{y=-2} \\ 0 & \underline{y<-2} \\ & \text{(else)} \end{cases}$$

$F_Y(y)$ has a jump at $y=2$ of height $F_Y(2)-F_Y(2_-) = \frac{1}{2}e^{-2}$

$F_Y(y)$ - — $y=0$ - - $F_Y(0)-F_Y(0_-) = 1-e^{-1}$

$F_Y(y)$ - - — $y=-2$ - - $F_Y(-2)-F_Y(-2_-) = \frac{1}{2}e^{-2}$

For $0<y<2$, $f_Y(y)=\frac{d}{dy}F_Y(y) = \frac{1}{4}e^{-\frac{1}{2}y-1}$

For $-2<y<0$. $f_Y(y)=\frac{d}{dy}F_Y(y) = \frac{1}{4}e^{\frac{1}{2}y-1}$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{2}e^{-2}\delta(y-2) & y=2 \\ \frac{1}{4}e^{-\frac{1}{2}y-1} & 0<y<2 \\ (1-e^{-1})\delta(y-0) & y=0 \\ \frac{1}{4}e^{\frac{1}{2}y-1} & -2<y<0 \\ \frac{1}{2}e^{-2}\delta(y+2) & y=-2 \\ 0 & y<-2 \text{ or } y>2 \end{cases}$$

(d) $E(Y) = \frac{1}{2}e^{-2}\times 2 + \int_0^2 \frac{1}{4}e^{-\frac{y+2}{2}} y\,dy + (1-e^{-1})\times 0 + \int_{-2}^0 \frac{1}{4}e^{\frac{1}{2}y-1} y\,dy$

$+ \frac{1}{2}e^{-2}\times(-2) = 0.$

$E(Y^2) = \frac{1}{2}e^{-2}\times 4 + \int_0^2 \frac{1}{4}e^{-\frac{y+2}{2}} y^2\,dy + (1-e^{-1})\times 0 + \int_{-2}^0 \frac{1}{4}e^{+\frac{1}{2}y-1} y^2\,dy$

$+\frac{1}{2}e^{-2}\times 4$

$=-16e^{-2}+8e^{-1}$

$Var(Y) = E[Y^2] - E^2[Y] = -16e^{-2}+8e^{-1}$

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4. **Text Problem 3.29 (page 133)** Let I be the index of the first successful refresh. Recall from Problem 3.18 that the PMF for I is

$$p_I(i) = (1/2)^i.$$

We may then compute the expected value of I from the definition:

$$E[I] = \sum_{i=1}^{\infty} i p_I(i) = \sum_{i=1}^{\infty} i(1/2)^i,$$

where this sum may be computed by a tweek to the geometric series: Recall the geometric series: $\sum_{i=m}^{n} \alpha^i = \frac{\alpha^m - \alpha^{n+1}}{1-\alpha}$. Therefore

$$\begin{aligned}
\frac{d}{d\alpha}\sum_{i=m}^{n} \alpha^i &= \frac{d}{d\alpha}\left[\frac{\alpha^m - \alpha^{n+1}}{1-\alpha}\right] \\
\Rightarrow \sum_{i=m}^{n} i\alpha^{i-1} &= \frac{[m\alpha^{m-1} - (n+1)\alpha^n][1-\alpha] + [\alpha^m - \alpha^{n+1}]}{(1-\alpha)^2} \\
&= \frac{m\alpha^{m-1} - (n+1)\alpha^n + (1-m)\alpha^m + n\alpha^{n+1}}{(1-\alpha)^2}
\end{aligned}$$

Since $\sum_{i=m}^{n} i\alpha^{i-1} = \frac{1}{\alpha}\sum_{i=m}^{n} i\alpha^i$, we arrive at

$$\sum_{i=m}^{n} i\alpha^i = \frac{m\alpha^m - (n+1)\alpha^{n+1} + (1-m)\alpha^{m+1} + n\alpha^{n+2}}{(1-\alpha)^2}$$

By this identity, we now have that

$$E[I] = \sum_{i=1}^{\infty} i(1/2)^i = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$$

To compute $Var[I]$ we first compute $E[I^2] = \sum_{i=1}^{\infty} i^2(\frac{1}{2})^i$. Similar to above, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} i^2\alpha^i = \alpha\frac{d}{di}\sum_{i=1}^{\infty} i\alpha^i &= \alpha\frac{d}{d\alpha}\left\{\frac{\alpha}{(1-\alpha)^2}\right\} \\
&= \alpha\left\{\frac{(1-\alpha)^2 + 2\alpha(1-\alpha)}{(1-\alpha)^4}\right\} \\
&= \frac{\alpha + \alpha^2}{(1-\alpha)^3}.
\end{aligned}$$

Therefore

$$\begin{aligned} E[I^2] &= \frac{\frac{1}{2} + (\frac{1}{2})^2}{(1-\frac{1}{2})^3} = 6 \\ \Rightarrow Var[I] &= E[I^2] - (E[I])^2 = 6 - 4 = 2. \end{aligned}$$

If we relate the *time* T, in seconds, it takes to renew to the *number* I of requests it takes to renew, we have that $T = 10I$, so that $E[T] = E[10I] = 10E[I] = 20$, and $Var[T] = Var[10I] = 100Var[I] = 200$.

5. **Text Problem 4.39 (page 219)** We have that the pdf $f_X(x) = c(1-x^2)$ We first have that constant c must be such that $\int_{-\infty}^{\infty} f_X(x)dx = 1$. Thus,

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-1}^{1} c(1-x^2)dx = c\left[x - \frac{1}{3}x^3\right]_{x=-1}^{1} = \frac{4c}{3},$$

therefore, $c = \frac{3}{4}$. Now the expected value

$$\begin{aligned} E[X] = \int_{-\infty}^{\infty} x f_X(x)dx &= \int_{-1}^{1} \frac{3}{4}x(1-x^2)dx \\ &= \frac{3}{4}\int_{-1}^{1} x - x^3 dx = \frac{3}{4}\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{x=-1}^{1} \\ &= 0 \end{aligned}$$

We compute the variance, $Var[X] = E[X^2] - (E[X])^2$ by first computing

$$\begin{aligned} E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx &= \int_{-1}^{1} \frac{3}{4}x^2(1-x^2)dx \\ &= \frac{3}{4}\int_{-1}^{1} x^2 - x^4 dx = \frac{3}{4}\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{x=-1}^{1} \\ &= \frac{4}{15} \end{aligned}$$

Therefore $Var[X] = E[X^2] - (E[X])^2 = \frac{4}{15} - 0 = \frac{4}{15}$.

6. **Text Problem 4.56 (page 220)**

a. Since $Y = 3X + 2$, we have that

$$\begin{aligned} E[Y] &= E[3X+2] = E[3X] + E[2] = 3E[X] + 2 \\ \text{and } Var[Y] &= Var[3X+2] = Var[3X] = 9Var[X], \end{aligned}$$

so that we need only know $E[X]$ and $Var[X]$ to compute $E[Y]$ and $Var[Y]$.

b. When X is Laplacian, we have $E[X] = 0$ and $Var[X] = 2/\alpha^2$, for parameter α. Thus, $E[Y] = 3E[X] + 2 = 2$ and $Var[Y] = 9Var[X] = 18/\alpha^2$.

c. When $X \sim \mathcal{N}(\mu, \sigma^2)$ (notation for X Gaussian random variable with mean μ and variance σ^2), we have $E[Y] = 3E[X]+2 = 3\mu+2$ and $Var[Y] = 9Var[X] = 9\sigma^2$.

d. If $X = b\cos(2\pi U)$, then we first find that

$$\begin{aligned} E[X] = E[b\cos(2\pi U)] &= \int_{-\infty}^{\infty} b\cos(2\pi u) f_U(u)du \\ &= \int_0^1 b\cos(2\pi u)du \\ &= \frac{b}{2\pi}\left[\sin(2\pi u)\right]_{u=0}^1 = 0 \\ \text{and } E[X^2] = E[b^2\cos^2(2\pi U)] &= \int_{-\infty}^{\infty} b^2\cos^2(2\pi u) f_U(u)du \\ &= b^2 \int_0^1 \cos^2(2\pi u)du \\ &= \frac{b^2}{2\pi}\left[\pi u + \frac{1}{4}\sin(4\pi u)\right]_{u=0}^1 = \frac{b^2}{2}. \end{aligned}$$

Therefore $E[X] = 0$ and $Var[X] = E[X^2] - (E[X])^2 = b^2/2$ so that $E[Y] = 3E[X] + 2 = 2$ and $Var[Y] = 9Var[X] = \frac{9b^2}{2}$.

7. (a) Ture by definition

(b) Ture by definition and the methods taught in class.

(c) False Although X is continuous r.v., Y might be discrete or mixed r.v.

(d) Ture. $E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx = E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy$.

(e) Ture $E[aX+b] = E[aX] + E[b] = aE[X] + b$

(f) Ture
$$\begin{aligned} Var[aX+b] &= E[(aX+b)^2] - E^2[aX+b] \\ &= E[a^2X^2 + 2abX + b^2] - a^2E^2[X] - 2abE[X] - b^2 \\ &= a^2E[X^2] - a^2E^2[X] \\ &= a^2\left(E[X^2] - E^2[X]\right) \\ &= a^2 Var[X] \end{aligned}$$

(g) False for every constant C $Var[C] = 0$.