1. Problem 3.41.

(a). The first draw is k dollars. $k \in \{1, 50\}$.

$Pr(k=1) = \frac{9}{10}$, $Pr(k=50) = \frac{1}{10}$

$Pr(X=51 \mid k=1) = \frac{1}{9}$, $Pr(X=2 \mid k=1) = \frac{8}{9}$

$Pr(X=51 \mid k=50) = 1$ $Pr(X=2 \mid k=50) = 0$

(b) $E[X \mid k=1] = \sum_i x_i Pr(X=x_i \mid k=1) = 51 \times \frac{1}{9} + 2 \times \frac{8}{9} = \frac{67}{9}$

$E[X \mid k=50] = \sum_i x_i Pr(X=x_i \mid k=50) = 51 \times 1 + 2 \times 0 = 51$

(c) $E[X] = \sum_m E[X \mid k=m] = \frac{67}{9} \times \frac{9}{10} + 51 \times \frac{1}{10} = \frac{118}{10} = 11.8$

(d) $E[X^2 \mid k=1] = \sum_i x_i^2 Pr(x_i \mid k=1) = 51^2 \times \frac{1}{9} + 2^2 \times \frac{8}{9} = \frac{2633}{9}$

$E[X^2 \mid k=50] = \sum_i x_i^2 Pr(x_i \mid k=1) = 51^2 \times 1 + 2^2 \times 0 = 2601$

$E[X^2] = \sum_m E[X^2 \mid k=m] Pr(k=m) = \frac{2633}{9} \times \frac{9}{10} + 2601 \times \frac{1}{10} = 523.4$

$Var[X] = E[X^2] - E^2[X] = 384.16$.

2. Problem 4.32

$$F_X(x) = \begin{cases} 0 & \text{for } x<0 \\ 1-\frac{1}{4}e^{-2x} & x \geq 0 \end{cases}$$

There is a jump at $x=0$, $F_X(0) - F_X(0^-) = \frac{3}{4}$

For $x>0$, $f_X(x) = \frac{dF_X(x)}{dx} = \frac{1}{2}e^{-2x}$

$$\therefore f_X(x) = \begin{cases} \frac{1}{2}e^{-2x} & x>0 \\ \frac{3}{4}\delta(x) & x=0 \\ 0 & \text{else} \end{cases}$$

$\Pr(B) = \Pr(\{X>0.25\}) = 1 - F_X(0.25) = 0.25\,e^{-0.5}$

$$F_X(x|B) = F_X(x|X>0.25) = \Pr(X \leq x \,|\, X>0.25)$$

$$= \frac{\Pr(\{X\leq x\} \cap \{X>0.25\})}{\Pr(\{X \geq 0.25\})}$$

$$= \begin{cases} 0 &, x \leq 0.25 \\ \dfrac{F_X(x) - F_X(0.25)}{0.25e^{-0.5}} &, x>0.25 \end{cases}$$

$$= \begin{cases} 1-e^{0.5-2x} &, x>0.25 \\ 0 & x\leq 0.25 \end{cases}$$

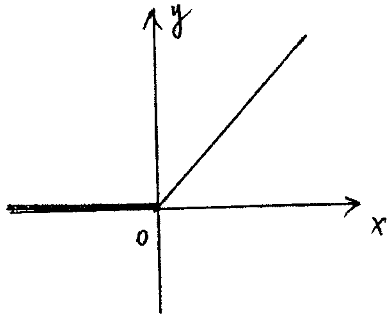
$$f_X(x|B) = \frac{dF_X(x|B)}{dx} = \begin{cases} 2e^{0.5-2x} &, x>0.25 \\ 0 &, x\leq 0.25 \end{cases}$$

$$E[X|B] = \int_{0.25}^{\infty} x f_X(x|B)\, dx = \int_{0.25}^{\infty} 2x e^{0.5-2x}\, dx = \frac{3}{4}$$

$$E[X^2|B] = \int_{0.25}^{\infty} x^2 f_X(x|B)\, dx = \int_{0.25}^{\infty} 2x^2 e^{0.5-2x}\, dx = \frac{13}{16}$$

$$Var[X|B] = E[X^2|B] - E^2[X|B] = \frac{13}{16} - \left(\frac{3}{4}\right)^2 = \frac{1}{4}$$

3. Input $X \sim N(0, \sigma^2)$



(a) current $I = \frac{Y}{R}$

$P_r(\{\frac{Y}{R} > 1\}) = P_r(\{Y > R\}) = P_r(\{X > R\})$ $\quad (R > 0)$

$P_r(\{X > R\}) = \int_R^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = \boxed{Q(\frac{R}{\sigma})}$

(b) Power $P = \frac{Y^2}{R}$

$P_r(\{\frac{Y^2}{R} > 1\}) = P_r(\{Y > \sqrt{R}\}) = P_r(\{X > \sqrt{R}\})$. $\quad (R > 0)$

$P_r(\{X > \sqrt{R}\}) = \boxed{Q(\frac{\sqrt{R}}{\sigma})}$

(c) $E[\frac{Y}{R}] = E[\frac{1}{R} X u(X)] = \frac{1}{R} \int_{-\infty}^{\infty} x u(x) f_X(x) dx = \frac{1}{R} \int_0^\infty x f_X(x) dx$

$= \frac{1}{R} \int_0^\infty x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma R} (-\sigma^2) e^{-\frac{x^2}{2\sigma^2}} \Big|_0^\infty = \boxed{\frac{\sigma}{\sqrt{2\pi} R}}$

$E[\frac{Y^2}{R^2}] = \frac{1}{R^2} E[X^2 u(X)] = \frac{1}{R^2} \int_{-\infty}^{\infty} x^2 u(x) f_X(x) dx = \frac{1}{R^2} \int_0^\infty x^2 f_X(x) dx$

$= \frac{1}{R^2} \int_0^\infty x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma R^2} \left\{ \left[-x\sigma^2 e^{-\frac{x^2}{2\sigma^2}} \Big|_0^\infty \right] + \sigma^2 \int_0^\infty e^{-\frac{x^2}{2\sigma^2}} dx \right\}$

$= \frac{\sigma}{\sqrt{2\pi} R^2} \left(\int_0^\infty e^{-\frac{x^2}{2\sigma^2}} dx \right)$

Suppose $A = \int_0^\infty e^{-\frac{x^2}{2\sigma^2}} dx$

$A^2 = A \cdot A = \int_0^\infty e^{-\frac{x^2}{2\sigma^2}} dx \cdot \int_0^\infty e^{-\frac{y^2}{2\sigma^2}} dy = \int_0^\infty \int_0^\infty e^{-\frac{x^2+y^2}{2\sigma^2}} dx dy$

$\xrightarrow[tg\theta = \frac{y}{x}]{\text{let } x^2+y^2=r^2} \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-\frac{r^2}{2\sigma^2}} r \, d\theta \, dr = \frac{\sigma^2 \pi}{2}$

Hence, $E[\frac{Y^2}{R^2}] = \frac{\sigma}{\sqrt{2\pi}R^2} A = \frac{\sigma^2}{2R^2}$

$$\sigma_I^2 = Var[\frac{Y}{R}] = E[\frac{Y^2}{R^2}] - E^2[\frac{Y}{R}] = \boxed{\frac{\sigma^2}{2R^2} - \frac{\sigma^2}{2\pi R^2}}$$

(d) $E[\frac{Y^2}{R}] = \frac{1}{R} E[Y^2] = \frac{1}{R}\int_{-\infty}^{\infty} x^2 u(x) f_X(x)\,dx = \frac{1}{R}\int_0^{\infty} x^2 f_X(x)\,dx$

$$= \frac{1}{R}\int_0^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx = \boxed{\frac{\sigma^2}{2R}}$$

From $E[(X-\bar{X})^4] = 3\sigma^4$ and $\bar{X} = 0$

Hence, $\int_{-\infty}^{\infty} x^4 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx = 3\sigma^4$

$$\int_0^{\infty} x^4 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} x^4 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx = \frac{3}{2}\sigma^4$$

$$E[\frac{Y^4}{R^2}] = \frac{1}{R^2}E[Y^4] = \frac{1}{R^2}\int_{-\infty}^{\infty} x^4 u(x) f_X(x)\,dx = \frac{1}{R^2}\int_0^{\infty} x^4 f_X(x)\,dx = \frac{1}{R^2}\int_0^{\infty} x^4 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$= \frac{3\sigma^4}{2R^2}$$

$$\sigma_P^2 = Var[\frac{Y^2}{R}] = E[\frac{Y^4}{R^2}] - E^2[\frac{Y^2}{R}] = \frac{3\sigma^4}{2R^2} - \frac{\sigma^4}{4R^2} = \boxed{\frac{5\sigma^4}{4R^2}}$$

to (c), when we want to find σ_I^2

The alternative method is:

we know that $Var[X] = \sigma^2$ (X is Gaussian Random Variable)

Hence, $E[X^2] = Var[X] + E^2[X] = \sigma^2 + 0 = \sigma^2$

$$E[Y^2] = \int_0^{\infty} x^2 f_X(x)\,dx = \frac{1}{2}E[X^2] = \frac{1}{2}\int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \frac{1}{2}\sigma^2$$

$$\sigma_I^2 = Var[\frac{Y}{R}] = E[\frac{Y^2}{R^2}] - E^2[\frac{Y}{R}] = \frac{1}{R^2}E[Y^2] - E^2[\frac{Y}{R}] = \boxed{\frac{\sigma^2}{2R^2} - \frac{\sigma^2}{2\pi R^2}}$$

4. (text problem 4.68 p. 221)

We have the lifetime of chip 1 $L_1 \sim \mathcal{N}(20000, 5000)$, while that of chip 2 is $L_2 \sim \mathcal{N}(22000, 1000)$. Our target lifetime is L_T, so we wish to compare L_1 and L_2 in terms of the probability that the lifetime is at least 24000; i.e. we declare chip 1 is superior if

$$P[L_1 \geq L_T] > P[L_2 \geq L_T]$$

otherwise we declare chip 2 is superior. If the target lifetime $L_T = 20000$, we find

$$\begin{aligned}
P[L_1 \geq 20000] &= 1 - P[L_1 < 20000] \\
&= 1 - \Phi\left(\frac{(20000 - 20000)}{5000}\right) \\
&= 1 - \Phi(0) = Q(0) = \frac{1}{2}, \\
\text{and} P[L_2 \geq 20000] &= 1 - P[L_2 < 20000] \\
&= 1 - \Phi\left(\frac{(20000 - 22000)}{1000}\right) \\
&= 1 - \Phi(-2) = \Phi(2) = 1 - Q(2) \approx 0.9772
\end{aligned}$$

Therefore chip 2 is superior in this case. Now let the target lifetime $L_T = 22000$, we find similarly

$$\begin{aligned}
P[L_1 \geq 22000] &= 1 - P[L_1 < 22000] \\
&= 1 - \Phi\left(\frac{(20000 - 22000)}{5000}\right) \\
&= 1 - \Phi(-0.4) = 1 - Q(0.4) \approx 0.6554, \\
\text{and} P[L_2 \geq 22000] &= 1 - P[L_2 < 20000] \\
&= 1 - \Phi\left(\frac{(22000 - 22000)}{1000}\right) \\
&= 1 - \Phi(0) = \frac{1}{2}.
\end{aligned}$$

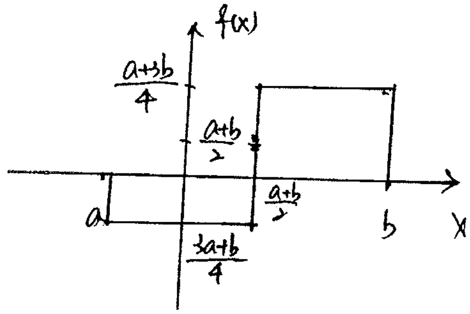
Therefore chip 1 is superior in this case.

5. (a) If the quantizer has k levels, then the ith interval will be.

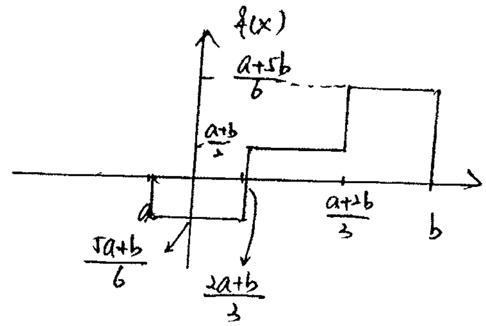
$x \in \left(a + (i-1) \times \frac{b-a}{k},\ a + i \times \frac{b-a}{k}\right),\ i = 1, \dots, k$

when x is in the interval, the output $Y_i = f(x)$ is equal to the midpoint of the interval.

$$y = a + \frac{(b-a)(2i-1)}{2k}$$



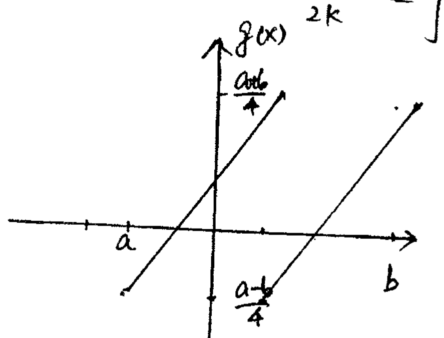
for $k=2$



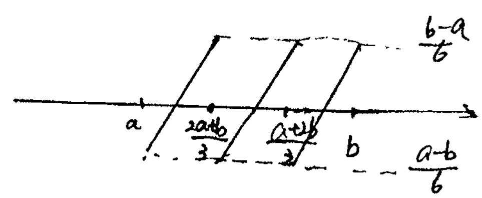
for $k=3$

(b) $g(x) = x - f(x)$

$$g(x) = x - \left[a + \frac{(b-a)(2i-1)}{2k}\right] \qquad x \in \left(a + (i-1) \times \frac{b-a}{k},\ a + i \times \frac{b-a}{k}\right)\ i = 1, \dots k$$



for $k=2$



(c) $Y = f(X)$ $\quad Y$ is a discrete random variable. $f_Y(y) = \sum_{i=1}^{k} \Pr\left(Y = a + \frac{(b-a)(2i-1)}{2k}\right) \delta\left[y - \left[a + \frac{(b-a)(2i-1)}{2k}\right]\right]$

$$\boxed{f_Y(y) = \sum_{i=1}^{k} \frac{1}{k}\, \delta\left(y - \left[a + \frac{(b-a)(2i-1)}{2k}\right]\right)} \qquad = \sum_{i=1}^{k} \underbrace{\Pr\left(a + (i-1) \times \frac{b-a}{k} < X < a + i \times \frac{b-a}{k}\right)}_{= \frac{1}{k}} \delta[\ \ ]$$

$E = g(X)$

$$f_E(e) = \sum_{i=1}^{k} f_X(x) \left|\frac{dx_i}{de}\right| \qquad \left(f_X(x) = \frac{1}{b-a};\ \frac{dx_i}{de} = 1,\ E = g(x) = x - \left[a + \frac{(b-a)(2i-1)}{2k}\right]\right)$$

$$\boxed{= \begin{cases} \frac{k}{b-a} & , E \in \left[\frac{a-b}{2k}, \frac{b-a}{2k}\right] \\ 0 & , \text{else} \end{cases}}$$

7

(d) $E[E^2] = \int_a^b g^2(x) f_X(x)\, dx = k \int_a^{a+\frac{1}{k}(b-a)} \left(x - \left(a + \frac{b-a}{2k}\right)\right)^2 \frac{1}{b-a}\, dx$

$$= \frac{k}{b-a} \frac{1}{3} \left[x - \left(\frac{(2k-1)a+b}{2k}\right)\right]^3 \Bigg|_a^{a+\frac{b-a}{k}}$$

$$= \frac{k}{3(b-a)} \left\{ \left[a + \frac{b-a}{k} - \left(\frac{(2k-1)a+b}{2k}\right)\right]^3 - \left[a - \left(\frac{(2k-1)a+b}{2k}\right)\right]^3 \right\}$$

$$= \frac{k}{3(b-a)} \left[\left(\frac{b-a}{2k}\right)^3 - \left(\frac{a-b}{2k}\right)^3\right]$$

$$= \frac{(b-a)^2}{12k^2}$$

6. (text problem 3.39 p. 134) We have that a successful transmission happens when only one of the two transmitters send. If $T_i(t)$ is the event that transmitter i transmits at time t, then the probability of a successful transmission at time t is

$$P[T_1 \cap \bar{T}_2] + P[\bar{T}_1 P[T_2] = P[T_1]P[\bar{T}_2] + P[\bar{T}_1]P[T_2] = \frac{1}{2}$$

When $X = k$, we have the first successful transmission at time $t = k$. This can only occur if the first $k-1$ transmissions fail, followed by the success at $t = k$. Therefore,

$$p_X(k) = P[X = k] = \left(1 - \frac{1}{2}\right)^{k-1} \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{k}$$

We can then compute the mean of X as

$$E[X] = \sum_{k=1}^{\infty} k p_X(k) = \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^{k} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$$

a. Knowing that the first transmission failed, we now consider the conditional distribution of X given that $X > 1$, i.e. we find $p_X(k|X > 1) = P[X = k|X > 1]$. Since we assume $X > 1$, we have $P[X = 1] = 0$. However, for $k > 1$, we have that $X = k$ when after the first failed transmission, $k - 2$ *more* transmissions fail followed by the first success. Therefore,

$$p_X(k|X > 1) = \begin{cases} 0, & k \leq 1 \\ \left(\frac{1}{2}\right)^{k-2}\left(1 - \frac{1}{2}\right) = \left(\frac{1}{2}\right)^{k-1}, & k > 1 \end{cases}$$

We then find the conditional expected value of X,

$$\begin{aligned} E[X|X > 1] = \sum_{k=1}^{\infty} k p_X(k|X > 1) &= \sum_{k=2}^{\infty} k \left(\frac{1}{2}\right)^{k-1} \\ (\text{let } l = k - 1) &= \sum_{l=1}^{\infty} (l+1) \left(\frac{1}{2}\right)^{l} \\ &= \sum_{l=1}^{\infty} l \left(\frac{1}{2}\right)^{l} + \sum_{l=1}^{\infty} \left(\frac{1}{2}\right)^{l} \\ &= E[X] + 1 = 3 \end{aligned}$$

Therefore, we have proved the identity.

b. Given that the first transmission succeeded, we know with assurance that the first successful transmission occurred at $t = 1$. Therefore, $X = 1$ with probability 1. In this case $E[X|X = 1] = 1$.

c. We have

$$E[X] = E[X|X > 1]P[X > 1] + E[X|X = 1]P[X = 1].$$

Since $P[X = 1] = p_X(1) = \frac{1}{2}$, then $P[X > 1] = 1 - P[X = 1] = \frac{1}{2}$. Thus,

$$E[X] = 3/2 + 1/2 = 2$$

d. We first have that

$$E[X^2|X = 1] = E[1^2] = 1.$$

We then have that

$$Var[X|X = 1] = E[X^2|X = 1] - (E[X|X = 1])^2 = 1 - 1^2 = 0.$$

This makes sense, since X is a constant, given that $X = 1$.
Now, given that the first transmission fails, $X > 1$ so that

$$\begin{aligned}
E[X^2|X > 1] = \sum_{k=2}^{\infty} k^2 p_X(k|X > 1) &= \sum_{k=2}^{\infty} k^2 \left(\frac{1}{2}\right)^{k-1} \\
&= 2\sum_{k=2}^{\infty} k^2 \left(\frac{1}{2}\right)^{k} \\
&= 2\left[-\frac{1}{2} + \sum_{k=1}^{\infty} k^2 \left(\frac{1}{2}\right)^{k}\right] \\
&= -1 + 2\left[\frac{\frac{1}{2} + \left(\frac{1}{2}\right)^2}{(1 - \frac{1}{2})^2}\right] \\
&= -1 + 2 \times 6 = 11
\end{aligned}$$

Therefore,

$$Var[X|X > 1] = E[X^2|X > 1] - (E[X|X > 1])^2 = 11 - 3^2 = 2.$$

We have, now, that

$$\begin{aligned}
E[X^2] &= E[X^2|X = 1]P[X = 1] + E[X^2|X > 1]P[X > 1] = (1)\tfrac{1}{2} + (11)\tfrac{1}{2} = 6, \\
\Rightarrow Var[X] &= E[X^2] - (E[X])^2 = 6 - 4 = 2.
\end{aligned}$$

7 We need to find the probability that the 7th passenger comes after 10 mins.

Let T_7 be a 7th order Erlang Random Variable.

$$Pr(T_7 > 10) = 1 - P(T_7 \le 10) \qquad \lambda = 1.$$

$$= 1 - F_{T_7}(10) = 1 - \left(1 - \sum_{n=0}^{6} \frac{(10)^n e^{-10}}{n!}\right)$$

$$= \sum_{n=0}^{6} \frac{(10)^n e^{-10}}{n!}$$

$$= \frac{10^0 e^{-10}}{0!} + \frac{10^1 e^{-10}}{1!} + \frac{10^2 e^{-10}}{2!} + \frac{10^3 e^{-10}}{3!} + \frac{10^4 e^{-10}}{4!} + \frac{10^5 e^{-10}}{5!} + \frac{10^6 e^{-10}}{6!}$$

$$= \frac{25799}{9e^{10}} = 0.13$$