Solutions

EE302 Midterm #1

MWF 11:30 - 12:20, Prof. Gelfand

Instructions:

- **There are 10 true-false problems (5 pts each) and 2 work-out problems (25 pts each). Do all problems**
- **You must show work to receive any credit on work-out problems.**
- **Calculators but not laptops are allowed**
- **Cheating will result in failure of the course. Do not cheat!**
- **Put your name on every page of the exam and turn in everything when time is called.**

Useful formula:

- Bernouilli trials: Pr (k successes in n trials) $= \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, \ldots, n$, where p is probability of success on a given trial. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, $0! = 1$.

Questions 1 - 10 are true-false problems (5 pts each). Label each statement true or false to the left of the problem number. (note: if statement is not always true, then it is false).

T 1. Let A, B and C be sets. Then $A - (B \cap C) = \left(A \cap \overline{B}\right) \cup \left(A \cap \overline{C}\right)$.

F 2. Let $A_1, \ldots, A_n$ be mutually exclusive events (with $\Pr(A_i) > 0$ for all i), and B be another event. Then $\Pr(B) \le \Pr(B|A_1)\Pr(A_1) + \cdots + \Pr(B|A_n)\Pr(A_n)$.

T 3. Let A, B and C be independent events. Then $\Pr\left((A \cup B) \cap \overline{C}\right) = \Pr(A \cup B)\Pr\left(\overline{C}\right)$.

F 4. Let X be a discrete random variable with pmf $p_X(x_i)$. Then $F_X(x_i) = p_X(x_i)$

T 5. Let X be a continuous random variable with pdf $f_X(x)$. Then $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$.

T 6. Let X be a discrete random variable and $Y = g(X)$, where $g(x)$ is an arbitrary function of x. Then Y is a discrete random variable.

F 7. Let X be a continuous random variable and $Y = g(X)$, where $g(x)$ is a smooth function of x. Then Y is a continuous random variable.

T 8. Let X be a continuous random variable and $Y = g(X)$, where $g(\cdot)$ is a smooth function of x. Then $f_Y(y) = \frac{d}{dy}\int_{x:g(x)\le y} f_X(x)\,dx$.

F 9. Let X be a continuous random variable and $Y = g(X)$, where $g(x)$ is a smooth function of x. Then $f_Y(y) = \sum_{x_i:g(x_i)=y} f_X(x_i)\left|\frac{1}{g'(x_i)}\right|$

F 10. Let X be a random variable. Then $(\mathrm{E}[X])^2 = \mathrm{Var}[X] + \mathrm{E}[X^2]$

Notes:

1\. $A - (B \cap C) = A \cap \left(\overline{B \cap C}\right) = A \cap \left(\overline{B} \cup \overline{C}\right) = \left(A \cap \overline{B}\right) \cup \left(A \cap \overline{C}\right)$

2\. $\Pr(B) \ge \Pr(B|A_1)\Pr(A_1) + \cdots + \Pr(B|A_n)\Pr(A_n)$ (lecture)

3\. A, B, C independent $\Rightarrow$ $A \cup B$, $\overline{C}$ independent

4\. $F_X(x_i) = \sum_{x_j \le x_i} p_X(x_j)$

5\. Lecture

6\. Lecture

7\. $g(x) = 0 \Rightarrow Y = 0$, discrete r.v.

8\. $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\int_{g(x)\le y} f_X(x)\,dx$ as in distribution method

9\. $f_Y(y) = \sum_{x_i : g(x_i) = y} f_X(x_i)\frac{1}{|g'(x_i)|} + \sum_j \Pr(a_j \le X \le b_j)\,\delta(y - y_j)$ where $g(x) = y_j$ for all $x \in [a_j, b_j]$

↑ impulsive component

10\. $\mathrm{Var}[X] = E[X^2] - (E[X])^2$

Questions 11 and 12 are work-out problems (25 pts each). You must show work to receive credit.

11. In a digital communications systems, a sequence of bits (0 or 1) is transmitted, and due to noise the received bit is sometimes incorrect. Assume that the transmitted bits are independent, and the received bit depends only on the corresponding transmitted bit. Also assume that for any bit transmission, the probability a 1 is transmitted is p, the probability a 0 is received given a 1 is transmitted is α, and the probability a 1 is received given a 0 is transmitted is β.

    (a) Find the probability the transmitted bit is 0 given the received bit is 1 for a single transmission

    (b) Find the probability the transmitted bit is 1 given the received bit is 0 for a single transmission.

    (c) Find the probability of an error for a single transmission.

    (d) Find the probability of k errors in n transmissions.

    (e) Find the probability that both a 1 is transmitted and an error occurs k times in n transmissions.

$T_i, R_i \equiv$ i is transmitted, received $(i = 0$ or $1)$

$Pr(T_1) = p$, $Pr(R_0 \mid T_1) = \alpha$, $Pr(R_1 \mid T_0) = \beta$

(a) $Pr(T_0 \mid R_1) = \dfrac{Pr(R_1 \mid T_0)\, Pr(T_0)}{Pr(R_1)}$

$Pr(R_1) = Pr(R_1 \mid T_0)\, Pr(T_0) + Pr(R_1 \mid T_1)\, Pr(T_1) = \beta(1-p) + (1-\alpha)p$

$\therefore\ Pr(T_0 \mid R_1) = \dfrac{\beta(1-p)}{\beta(1-p) + (1-\alpha)p}$ //

(b) $Pr(T_1 \mid R_0) = \dfrac{Pr(R_0 \mid T_1)\, Pr(T_1)}{Pr(R_0)}$ $= 1 - Pr(R_1)$

$= \dfrac{\alpha p}{1 - [\beta(1-p) + (1-\alpha)p]}$ //

(c) $Pr(\text{Error}) = Pr((T_0 \cap R_1) \cup (T_1 \cap R_0))$

$= Pr(R_1 \mid T_0)\, Pr(T_0) + Pr(R_0 \mid T_1)\, Pr(T_1)$

$= \beta(1-p) + \alpha p$ //

(d) Bernoulli trials with prob. ~~p_E~~ $Pr(\text{Error}) = \beta(1-p) + \alpha p$

~~$\binom{n}{k} p_E^k (1-p_E)^{n-k}, k = 0, \ldots, n$~~

$Pr(k \text{ errors in } n \text{ transmissions}) = \binom{n}{k} [\beta(1-p) + \alpha p]^k [1 - [\beta(1-p) + \alpha p]]^{n-k}$

$k = 0, \ldots, n$ //

e) Bernoulli trials with prob.

$Pr(\text{Error and } 1 \text{ transmitted}) = Pr(\text{Error} \mid T_1)\, Pr(T_1)$

$= Pr(R_0 \mid T_1)\, Pr(T_1) = \alpha p$

$\therefore$ $Pr(\overset{\text{both}}{\wedge}\, 1$ transmitted and error occurs k times in n transmissions$)$ $= \binom{n}{k} (\alpha p)^k (1-\alpha p)^{n-k}$

$k = 0, \ldots, n$ //

12. Let X be a random variable and $Y = g(X)$, where

$$g(x) = \begin{cases} 1, & x \geq 0.5, \\ 2x, & 0 \leq x \leq 0.5, \\ 0, & x \leq 0. \end{cases}$$

(a) Plot $g(x)$. Is $g(x)$ smooth? piecewise smooth?

(b) Suppose X is a discrete random variable with pmf

$$p_X(x_i) = c, \quad \text{for } x_i = -1, 0, 1.$$

i. Find c.

ii. Find $p_Y(y_j)$.

iii. Find $\mathrm{E}[Y]$.

(c) Suppose X is a continuous random variable with pdf

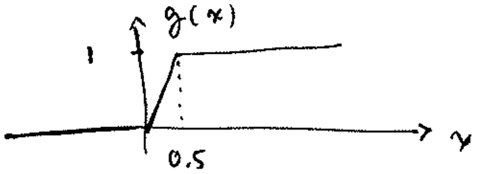
$$f_X(x) = \begin{cases} c, & -1 \leq x \leq 1, \\ 0, & \text{else.} \end{cases}$$

i. Find c.

ii. Find $f_Y(y)$. What kind of random variable is Y?

iii. Find $\mathrm{E}[Y]$.

(a)



piecewise smooth (but not smooth) since differentiable except at $x = 0, 0.5$

(b)(i) $\sum_{x_i} p_X(x_i) = \sum_{x_i=-1,0,1} c = 3c = 1 \Rightarrow c = \frac{1}{3}$ //

(ii) $p_Y(y_j) = \sum_{x_i : g(x_i) = y_j} p_X(x_i)$ $\quad x_i \in \{-1, 0, 1\} \Rightarrow y_j \in \{0, 1\}$

$p_Y(1) = p_X(1) = \frac{1}{3}$

$p_Y(0) = p_X(0) + p_X(-1) = \frac{2}{3}$ //

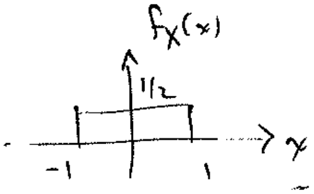
(iii) $E[Y] = \sum_{y_j} y_j \, p_Y(y_j) = 1\left(\frac{1}{3}\right) + 0\left(\frac{2}{3}\right) = \frac{1}{3}$ //

Alternatively

$E[Y] = E[g(X)] = \sum_{x_i} g(x_i) \, p_X(x_i)$

$= 0\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) = \frac{1}{3}$

(c) (i) $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-1}^{1} c\,dx = 2c = 1 \Rightarrow c = \frac{1}{2}$ //



(ii) $f_Y(y) = f_1(y) + f_2(y)$ //

$$f_1(y) = f_X(x)\left|\frac{dx}{dy}\right|, \quad 0 < y < 1$$
$$= 0, \quad \text{else}$$

When $y \in (0,1)$, $y = 2x \Rightarrow x = \frac{y}{2}$, $\left|\frac{dx}{dy}\right| = \frac{1}{2}$

$$\therefore f_1(y) = f_X\left(\frac{y}{2}\right)\cdot\frac{1}{2} \quad 0 < y < 1$$
$$= \frac{1}{4}, \quad 0 < y < 1 \quad //$$
$$= 0, \quad \text{else}$$

$$f_2(y) = Pr(Y=0)\,\delta(y) + Pr(Y=1)\,\delta(y-1)$$
$$= Pr(X \leq 0)\,\delta(y) + Pr(X \geqslant 0.5)\,\delta(y-1)$$
$$= \int_{-1}^{0} \frac{1}{2}\,dx\;\delta(y) + \int_{0.5}^{1} \frac{1}{2}\,dx \cdot \delta(y-1)$$
$$= \frac{1}{2}\delta(y) + \frac{1}{4}\delta(y-1) \quad //$$

Y is mixed r.v.
Since has both
impulsive & nonimpulsive
components in $f_Y(y)$

(iii) $E[Y] = \int y\, f_Y(y)\,dy$

$$= \int_0^1 y\cdot\frac{1}{4}\,dy + \int y\left(\frac{1}{2}\delta(y) + \frac{1}{4}\delta(y-1)\right)dy$$
$$= \frac{y^2}{8}\Big|_0^1 + \frac{1}{2}(0) + \frac{1}{4}(1) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8} \quad //$$

Alternatively

$$E[Y] = E[g(X)] = \int g(x)\, f_X(x)\,dx$$
$$= \int_{-1}^{0} 0\cdot\frac{1}{2}\,dx + \int_0^{0.5} 2x\cdot\frac{1}{2}\,dx + \int_{0.5}^{1} 1\cdot\frac{1}{2}\,dx$$
$$= \frac{x^2}{2}\Big|_0^{0.5} + \frac{x}{2}\Big|_{0.5}^{1} = \frac{0.25}{2} + \frac{1}{2} - \frac{0.5}{2} = \frac{3}{8}$$