

# Applications of Polytopic Approximations of Reachable Sets to Linear Dynamic Games and a Class of Nonlinear Systems<sup>1</sup>

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## Abstract

This paper presents applications of polytopic approximation methods for reachable set computation using dynamic optimization. The problem of computing exact reachable sets can be formulated in terms of a Hamilton-Jacobi partial differential equation (PDE). Numerical solutions which provide convergent approximations of this PDE have computational complexity which is exponential in the continuous variable dimension. Using dynamic optimization and polytopic approximation, computationally efficient algorithms for overapproximative reachability analysis have been developed for linear dynamical systems [1]. In this paper, we show that these can be extended to feedback linearizable nonlinear systems, linear dynamic games, and norm-bounded nonlinear systems. Three illustrative examples are presented.

## 1 Introduction

Reachability analysis for continuous and hybrid systems is important for the automatic verification of safety properties and for the synthesis of safe controllers for these systems [2, 3]. Convergent approximations of reachable sets for such systems can be computed by solving the exact Hamilton-Jacobi partial differential equation (PDE). Numerical methods have been devised to compute these convergent overapproximations [4, 5], which work well in up to three continuous variable dimensions, yet these methods are not practical for solving high dimensional problems. Therefore, approximate methods for reachable set computation have been proposed.

Tiwari and Khanna [6] and Alur et al. [7] propose predicate abstraction for reachable set computation: this method can be used to extract equivalent finite state models from complex, infinite state models, and then to use them to find approximate reachable sets of the original systems. In [8], Hwang et al. have used an augmented form of predicate abstraction to compute reachable sets for a simple biological cell network. However, since the accuracy of reachability analysis using predicate abstraction greatly depends on the choice of polynomials for abstraction, it is important to have information about a given system *a priori* (from analysis and simulations) to get good results in the reachability analysis. Chutinan and Krogh [9, 10] present a method to approximate the flows of autonomous systems with convex polyhedra. An experimental system called *d/dt* [7, 11, 12] has been developed to approximate reachable sets for linear dynamical systems using griddy orthogonal polyhedra. In these methods, it is difficult to compute the control input which is guaranteed to keep the system on the boundary or inside the set, from the boundary of the overapproximative set.

Varaiya [1] designs a polytopic approximation for linear time invariant systems using optimal control. Kostousova [13] develops two-sided approximations of reachable sets for linear dynamic systems using parallelotopes. Kurzhanski and Varaiya [14, 15] propose an ellipsoidal approximation for forward and backward reachable sets (a computational tool VeriSHIFT [16] has been developed based on their ideas) and in [17, 18], they define various types of reachable sets for linear time-varying systems with bounded perturbations using both open and closed-loop input laws. In [18], they propose ellipsoidal overapproximations of reachable sets for linear systems under uncertainty via solutions of a particular type of differential equation. The authors clearly state in the conclusions of both papers that the computational methods of reachable sets and their approximations are topics for future research. These statements were a motivation for the results pre-

<sup>1</sup>This research is supported by DARPA under the Software Enabled Control Program (AFRL contract F33615-99-C-3014), by ONR under MURI contract N00014-02-1-0720, and by an NSF Career Award (ECS-9985072).

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sented in this paper.

In this paper, we use the method proposed by Varaiya [1] to compute reachable sets for linear time invariant systems using dynamic optimization, and inspired by the work of Khrustalev [19] and Kurzbaniski and Varaiya [14, 15, 17, 18], we extend it to more general dynamical systems such as feedback linearizable nonlinear systems, linear dynamic games, and norm-bounded nonlinear systems. We present three examples, one of which is a “benchmark” two-aircraft three-dimensional collision avoidance example, which is solved using feedback linearization with dynamic extension.

This paper is organized as follows. Motivation for this study is described in Section 2. Computations of polytopic reachable sets for linear dynamical systems, feedback linearizable nonlinear systems, linear dynamic games, and norm-bounded nonlinear systems are presented in Section 3. Examples are presented in Section 4. Conclusions are presented in Section 5.

## 2 Background and Motivation

Consider a dynamical system,

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), d(t)), \\ x(0) &\in X_0 \quad (\text{or } x(t_f) \in \mathcal{Y}_0), \quad t \in [0, t_f] \end{aligned} \quad (1)$$

where  $0 \leq t_f < \infty$ ,  $x \in \mathbb{R}^n$ ,  $u \in U \subset \mathbb{R}^m$  is the control input,  $d \in D \subset \mathbb{R}^p$  is the disturbance input,  $X_0 = \{x : l(x) \leq 0\}$  is an initial set of states, and  $\mathcal{Y}_0 = \{x : y(x) \leq 0\}$  is a target set of states. We assume  $f$  to be Lipschitz. The spaces of admissible control input trajectories and disturbance input trajectories are denoted as the spaces of piecewise continuous functions  $\mathcal{U} = \{u(\cdot) \in PC^0 | u(t) \in U, 0 \leq t \leq t_f\}$  and  $\mathcal{D} = \{d(\cdot) \in PC^0 | d(t) \in D, 0 \leq t \leq t_f\}$  respectively. The forward and the backward reachable sets of the system (1) are defined as follows.

**Definition 1** *The forward reachable set  $\mathcal{X}(\tau)$  at time  $\tau$  ( $0 < \tau \leq t_f$ ), of the system (1) from the initial set  $X_0$ , is the set of all states  $x(\tau)$ , such that there exists a control input  $u(t) \in \mathcal{U}$  ( $0 \leq t \leq \tau$ ), for all disturbance inputs  $d(t) \in \mathcal{D}$  ( $0 \leq t \leq \tau$ ), for which  $x(\tau)$  is reachable from some  $x(0) \in X(0)$ , along a trajectory satisfying (1).*

**Definition 2** *The backward reachable set  $\mathcal{Y}(\tau)$  at time  $\tau$  ( $0 \leq \tau < t_f$ ), of the system (1) from the target set  $\mathcal{Y}_0$ , is the set of all states  $x(\tau)$ , such that there exists a control input  $u(t) \in \mathcal{U}$  ( $\tau \leq t \leq t_f$ ), for all disturbance inputs  $d(t) \in \mathcal{D}$  ( $\tau \leq t \leq t_f$ ), for which some  $x(t_f) \in \mathcal{Y}_0$  are reachable from  $x(\tau)$ , along a trajectory satisfying (1).*

It has been shown that a forward reachable set computation can be formulated as a dynamic optimization problem [15, 19]. The forward reachable set of the dynamical system (1) at time  $\tau$  ( $0 < \tau \leq t_f$ ) is shown to be [15]:

$$\mathcal{X}(\tau) = \{x : v(x, \tau) \leq 0\} \quad (2)$$

where  $v(x, \tau)$  is a (viscosity) solution of the Hamilton-Jacobi-Isaacs (HJI) partial differential equation,

$$D_t v(x, t) + \max_{u \in U} \min_{d \in D} \{ \langle D_x v(x, t), f(x, u, d) \rangle \} = 0 \quad (3)$$

with  $v(x, 0) = l(x)$  and  $\langle p, q \rangle = p^T q$  the inner product in  $\mathbb{R}^n$ . Thus, the forward reachable set of the dynamical system (1) is the zero sublevel set of the solution to the HJI equation in (3).

Similarly, the backward reachable set of the dynamical system (1) at time  $\tau$  ( $0 \leq \tau < t_f$ ) is a zero sublevel set of the solution to the HJI equation [15],

$$D_t v(x, t) + \min_{u \in U} \max_{d \in D} \{ \langle D_x v(x, t), f(x, u, d) \rangle \} = 0 \quad (4)$$

with  $v(x, t_f) = y(x)$ .

In [4, 5], a numerical tool for computing convergent approximations for backwards reachable sets is designed and presented. This method is based on the level set method for computing PDE solutions [20]. The computational complexity of this tool is exponential in the number of continuous variables dimensions: it has been shown to work well in up to three continuous variables dimensions, yet for larger problems computation time is currently prohibitive. Numerical convergence has been demonstrated on several examples; we will use a “benchmark” three-dimensional example from [5] in the current paper.

Consider planar kinematic models of two aircraft, labeled 1 and 2. Let  $(x_r, y_r, \psi_r) \in \mathbb{R}^2 \times [-\pi, \pi)$  represent the relative position and orientation of aircraft 2 with respect to aircraft 1. Given the absolute positions and orientations of the two aircraft, denoted as  $x_i, y_i, \psi_i$  for  $i = 1, 2$ , the relative coordinates are defined as:  $x_r = \cos \psi_1 (x_2 - x_1) + \sin \psi_1 (y_2 - y_1)$ ,  $y_r = -\sin \psi_1 (x_2 - x_1) + \cos \psi_1 (y_2 - y_1)$ ,  $\psi_r = \psi_2 - \psi_1$ . The relative kinematics are thus given by:

$$\begin{aligned} \dot{x}_r &= -\sigma_1 + \sigma_2 \cos \psi_r + \omega_1 y_r \\ \dot{y}_r &= \sigma_2 \sin \psi_r - \omega_1 x_r \\ \dot{\psi}_r &= \omega_2 - \omega_1 \end{aligned} \quad (5)$$

where  $\sigma_i$  is the linear velocity of aircraft  $i$  and  $\omega_i$  is its angular velocity. Safety is encoded as a 5 nautical mile radius cylinder “protected zone” centered at the origin of the relative frame. In this paper, following the notation in Definition 2 (which is different from that in [5]), we define the angular velocity of aircraft 2 ( $\omega_2$ ) as the control input that steers the system (5) into the target set and the angular velocity of aircraft 1 ( $\omega_1$ ) as the disturbance input that keeps the system (5) outside of the target set. Posing this problem as a game, we label aircraft 1 as “evader” and aircraft 2 as “pursuer”, and we compute the set of states  $(x_r, y_r, \psi_r)$  for which for all possible disturbance inputs,  $\omega_1$  action of the evader, there is a control input,  $\omega_2$  action of the pursuer, such that the system state enters the protected

zone, which we consider the target set of the game. For values  $\sigma_1 = \sigma_2 = 5$  and  $\omega_i \in [-1, 1]$  ( $i \in \{1, 2\}$ ), the problem has been solved numerically, and the results (solid surface) are shown in Figure 3 (Courtesy of I. Mitchell [5]). This computation took approximately 15 minutes to run on a Sun UltraSparc II, in which 50 grid nodes in each dimension were used.

In the following section, we extend Varaiya's method [1] to treat this kind of system and in Section 4, we compare the above computation with the resulting approximation.

### 3 Computation of polytopic reachable sets

We first define the overapproximate reachable set [15] (here we specialize to the case of (1) in which there are no disturbances). Assume that  $x_*(0) \in X_0$  and  $u_*(t) \in \mathcal{U}$  for all  $t \geq 0$  such that  $x_*(\tau) \in \mathcal{X}(\tau)$  ( $0 \leq t \leq \tau$ ). Then, an overapproximate solution to the solution of the HJI equation in (3) is defined as a function  $v^+(x, t)$  satisfying [15, 19]:

$$\begin{aligned} & \frac{dv^+(x, t)}{dt} \Big|_{x=x_*(t), u=u_*(t), \dot{x}=f(x, u)} \\ &= D_t v^+(x_*, t) + \langle D_x v^+(x_*, t), f(x_*, u_*) \rangle \\ &\leq D_t v^+(x_*, t) + \max_{u \in U} \{ \langle D_x v^+(x_*, t), f(x_*, u) \rangle \} \\ &\leq \mu(t) \end{aligned} \quad (6)$$

where  $v^+(x_*, t)$  is a piecewise continuous function, and  $\mu(t)$  is a positive-definite, integrable function. By integrating (6) from 0 to  $\tau$ , we obtain an overapproximate reachable set of the dynamical system (1) at time  $\tau$  as:

$$V^+(\tau) = \{x | v^+(x, \tau) \leq \int_0^\tau \mu(t) dt + \max_{x(0) \in X_0} v^+(x(0), 0)\} \quad (7)$$

Next, we review the polytopic overapproximation of reachable sets for linear dynamical systems and derive computational methods for polytopic overapproximate reachable sets for feedback linearizable nonlinear systems, linear dynamic games, and norm-bounded nonlinear systems.

#### 3.1 Linear dynamical systems

In this section, we review the polytopic overapproximation of reachable sets for linear systems from [1]. Consider a time-varying linear dynamical system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) \in X_0, \quad u(t) \in U \quad (8)$$

where the initial set  $X_0$  and the admissible control input set  $U$  are assumed to be convex polytopes which have  $N$  and  $N_u$  faces respectively. In this paper, we assume the initial set  $X_0$  is a polytope, but in general the number of faces of the initial set is a design parameter since  $X_0$  may be a convex compact set and thus the more the number of faces of  $X_0$  the better the overapproximate reachable sets.

A convex polytope  $\mathcal{P}$  with  $K$  faces can be represented in two ways; it can be represented as the

bounded intersection of  $K$  half spaces,

$$\mathcal{P} = \bigcap_{i=1}^K \{x | h_i^T x \leq \gamma_i\} \quad (9)$$

where  $h_i$  is a normal vector to the  $i^{\text{th}}$  face of the polytope  $\mathcal{P}$ . A convex polytope can also be represented as the convex hull of its vertices: if a convex polytope  $\mathcal{P}$  has  $m$  vertices  $\{v^1, \dots, v^m\}$ , then

$$\mathcal{P} = \{x | x = \sum_{i=1}^m \alpha_i v^i, \alpha_i \geq 0, \sum_{i=1}^m \alpha_i = 1\} \quad (10)$$

Define a set of linear functions as

$$v_i^+(x, t) = h_i^T(t)x, \quad i \in \{1, 2, \dots, N\} \quad (11)$$

These linear functions are used to represent a convex polytope as shown in (9). In order to find a polytopic overapproximate reachable set, we solve for  $v_i^+(x, t)$  in (11) that satisfies (6). Then, (6) becomes

$$\begin{aligned} & D_t v_i^+(x, t) + \max_{u \in U} \{ \langle D_x v_i^+(x, t), f(x, u) \rangle \} \\ &= \langle \dot{h}_i(t), x(t) \rangle + \langle A(t)^T h_i(t), x(t) \rangle \\ &\quad + \max_{u \in U} \{ \langle h_i(t), B(t)u(t) \rangle \} \leq \mu(t) \end{aligned} \quad (12)$$

From optimal control theory [21], the adjoint equation for linear systems when the input set does not depend on  $x$  is  $\dot{\lambda}(t) = -A(t)^T \lambda(t)$ . If we choose  $h_i(t) = \lambda(t)$  ( $i \in \{1, 2, \dots, N\}$ ), then

$$\langle \dot{h}_i(t), x(t) \rangle + \langle A(t)^T h_i(t), x(t) \rangle = 0 \quad (13)$$

This represents the evolution of the normal vector of the  $i^{\text{th}}$  face. Let  $h_i(0)$ ,  $i \in \{1, 2, \dots, N\}$  be the normal vectors of the faces of the initial set  $X_0$ . Then, the solution to (13) is

$$h_i(t) = \Phi(t, 0)h_i(0), \quad i \in \{1, 2, \dots, N\} \quad (14)$$

where  $\Phi(t, 0)$  is the state transition matrix satisfying  $\dot{\Phi} = -A(t)^T \Phi$ ,  $\Phi(0, 0) = I$ . If the system dynamics in (8) is time invariant, then  $\Phi(t, 0) = e^{-A^T t}$  and (14) becomes

$$h_i(t) = e^{-A^T t} h_i(0), \quad i \in \{1, 2, \dots, N\} \quad (15)$$

Thus, for a linear time invariant system, the evolution of normal vectors can be determined analytically. We denote  $\{u^1, \dots, u^{m_u}\}$  as the vertices of the input set  $U$ . Since  $U$  is a convex polytope, the following must hold: (for  $j \in \{1, \dots, m_u\}$ )

$$\max_{u \in U} \langle h_i(t), B(t)u(t) \rangle = \max_j \langle h_i(t), B(t)u^j \rangle \leq \mu(t) \quad (16)$$

that is, the maximum is achieved at a vertex of  $U$  [1]. Furthermore, if the system dynamics in (8) is time invariant, (16) is simplified to

$$\max_j \langle h_i(t), B u^j \rangle = \max_j \langle e^{-A^T t} h_i(0), B u^j \rangle \leq \mu(t) \quad (17)$$

for  $j \in \{1, \dots, m_u\}$ . We choose  $\mu(t) = \max_j < h_i(t), B(t)u^j >$  and note that  $\mu(t)$  is always positive for a properly chosen input set  $U$  (e.g., chosen such that  $0 \in U$ ). Then, the linear function  $v_i^+(x, t)$  in (11) is a supporting hyperplane of the exact reachable set [1]. A polytopic overapproximate forward reachable set  $V^+(t)$  for the dynamical system (8) is the intersection of half spaces as follows:

$$V^+(t) = \bigcap_{i=1}^N \{x : v_i^+(x, t) \leq \int_0^t \max_j < h_i(s), B(s)u^j > ds + \max_{x(0) \in X_0} v_i^+(x(0), 0)\} \quad (18)$$

$V^+(t)$  is a convex polytope which contains the exact reachable set at time  $t$  since each  $v_i^+(x, t)$  in (18) is a supporting hyperplane of the exact reachable set. If the system dynamics is linear time invariant,  $V^+(t)$  becomes

$$V^+(t) = \bigcap_{i=1}^N \{x : v_i^+(x, t) \leq \int_0^t \max_j < e^{-A^T s} h_i(0), B u^j > ds + \max_{x(0) \in X_0} v_i^+(x(0), 0)\} \quad (19)$$

### 3.2 Feedback linearizable nonlinear systems

In this section, we consider a class of nonlinear systems [22], in which  $u(t)$  is a feedback control:

$$\dot{x}(t) = f(x) + g(x)u(t) \quad (20)$$

$$u(t) = a(x) + b(x)v(t) \quad (21)$$

We assume that there exists a diffeomorphism  $T$ : such that  $z = T(x)$ , which transforms, with a control input  $u(t)$ , a nonlinear system (20) into an equivalent linear system. Then, we can compute an overapproximate forward reachable set for the nonlinear system (20) as follows:

- Step 1: Transform the nonlinear system (20) to an equivalent linear system,  $\dot{z}(t) = A(t)z(t) + B(t)v(t)$  with appropriate  $u(t)$  and  $T$ .
- Step 2: Compute a polytopic overapproximate forward reachable set  $V^+(t)$  of the linear system following the procedure in Section 3.1.
- Step 3: Using the inverse state transformation  $x = T^{-1}(z)$ , we obtain the overapproximate forward reachable set for the original nonlinear system (20) from  $V^+(t)$ .

Since there is no approximation during the transformation and the transformation is a diffeomorphism on a given domain of interest, the forward reachable set obtained in Step 3 is guaranteed to be an overapproximate forward reachable set of the nonlinear system (20).

### 3.3 Linear dynamic games

Now, we consider the linear dynamic game:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)d(t), \quad x(0) \in X_0, \quad u(t) \in U, \quad d(t) \in D \quad (22)$$

where the initial set  $X_0$ , the admissible control input set  $U$ , and the disturbance input set  $D$  are assumed to be convex polytopes which have  $N$ ,  $N_u$ , and  $N_d$  faces respectively. Then, the HJI equation in (3) for a forward reachable set computation becomes [17, 18],

$$D_t v(x, t) + \max_{u \in U} \min_{d \in D} \{ < D_x v(x, t), A(t)x(t) + B(t)u(t) + C(t)d(t) > \} = 0 \quad (23)$$

To find an overapproximate solution to (23), we look for a set of linear functions  $v_i^+(x, t)$  in (11) satisfying (13), and compute

$$\begin{aligned} & D_t v_i^+(x, t) + \max_{u \in U} \min_{d \in D} \{ < D_x v_i^+(x, t), \\ & A(t)x(t) + B(t)u(t) + C(t)d(t) > \} \\ = & \max_{u \in U} \{ < h_i(t), B(t)u(t) > \} \\ & + \min_{d \in D} \{ < h_i(t), C(t)d(t) > \} \leq \mu(t) \end{aligned} \quad (24)$$

We denote  $\{u^1, \dots, u^{m_u}\}$  and  $\{d^1, \dots, d^{m_d}\}$  as the vertices of  $U$  and  $D$  respectively. Since (24) is linear with respect to  $u$  and  $d$ , the maximum and the minimum in (24) are achieved at vertices of  $U$  and  $D$  as follows:

$$\max_j < h_i(t), B(t)u^j > + \min_k < h_i(t), C(t)d^k > \leq \mu(t) \quad (25)$$

for  $j \in \{1, \dots, m_u\}$ ,  $k \in \{1, \dots, m_d\}$ .

We choose  $\mu(t) = \max_j < h_i(t), B(t)u^j > + \min_k < h_i(t), C(t)d^k >$  and then a polytopic overapproximate reachable set  $V^+(t)$  for a linear dynamic game (22) is

$$V^+(t) = \bigcap_{i=1}^N \{x : v_i^+(x, t) \leq \int_0^t \mu(s) ds + \max_{x(0) \in X_0} v_i^+(x(0), 0)\} \quad (26)$$

### 3.4 Norm-bounded nonlinear systems

In this section, we consider a norm-bounded nonlinear system,

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + \phi(x, t), \\ x(0) &\in X_0, \quad u(t) \in U, \quad \|\phi(x, t)\| \leq \beta(t) \end{aligned} \quad (27)$$

where the initial set  $X_0$  and the admissible control input set  $U$  are assumed to be convex polytopes which have  $N$  and  $N_u$  faces respectively and  $\|\cdot\|$  represents the Euclidean norm.  $\beta(\cdot)$  is a positive-definite function. Then, the HJI equation in (3) becomes

$$D_t v(x, t) + \max_{u \in U} \{ < D_x v(x, t), A(t)x(t) + B(t)u(t) + \phi(x, t) > \} = 0 \quad (28)$$

To compute an overapproximate solution to the HJB equation in (28), we find the linear functions  $v_i^+(x, t)$  in (11) satisfying (13), and compute

$$\begin{aligned} & D_t v_i^+(x, t) + \max_{u \in U} \{ < D_x v_i^+(x, t), \\ & A(t)x(t) + B(t)u(t) + \phi(x, t) > \} \\ = & \max_{u \in U} \{ < h_i(t), B(t)u(t) > \} + < h_i(t), \phi(x, t) > \\ \leq & \max_{u \in U} \{ < h_i(t), B(t)u(t) > \} + \frac{1}{2}(\|h_i(t)\|^2 + \|\phi(x, t)\|^2) \\ \leq & \max_j \{ < h_i(t), B(t)u^j > \} + \frac{1}{2}(\|h_i(t)\|^2 + \beta(t)^2) \leq \mu(t) \end{aligned} \quad (29)$$

If we choose  $\mu(t)$  such that

$$\mu(t) = \max_j \langle h_i(t), B(t)u^j \rangle + \frac{1}{2}(\|h_i(t)\|^2 + \beta(t)^2) \quad (30)$$

then, a polytopic overapproximate reachable set  $V^+(t)$  for the norm-bounded dynamical system (27) is

$$V^+(t) = \bigcap_{i=1}^N \{x : v_i^+(x, t) \leq \int_0^t [\max_j \langle h_i(s), B(s)u^j \rangle + \frac{1}{2}(\|h_i(s)\|^2 + \beta(s)^2)] ds + \max_{x(0) \in X_0} v_i^+(x(0), 0)\} \quad (31)$$

If  $\phi(x, t)$  belongs to a polytope with vertices  $\{\phi^1, \dots, \phi^{m_\phi}\}$ , a polytopic overapproximate reachable set  $V^+(t)$  becomes

$$V^+(t) = \bigcap_{i=1}^N \{x : v_i^+(x, t) \leq \int_0^t [\max_j \langle h_i(s), B(s)u^j \rangle + \max_k \langle h_i(s), \phi^k \rangle] ds + \max_{x(0) \in X_0} v_i^+(x(0), 0)\} \quad (32)$$

## 4 Examples

We consider three examples: a linear system, a norm-bounded nonlinear system, and we conclude with the example which motivated this study, a nonlinear, feedback linearizable, dynamic game. Note that equation (7) provides overapproximations of the sets of reachable states over a range of times (the flow). In the implementation, we compute overapproximations of the reachable sets at specific instants of time without interpolation between the sets.

### 4.1 Linear dynamical systems

In this section, we consider a linear dynamical system  $\dot{x} = Ax + Bu$ ,  $x(0) \in X_0$  where the control input  $u(t)$  can vary inside a convex polytope  $U$  and the initial set  $X_0$  is also a convex polytope. The system parameters ( $A, B, X_0$ , and  $U$ ) given in [11] are used. Figure 1 shows the evolution of the projection on  $x_3$  and  $x_4$  over time. This result is similar to that in [11], yet computation time with the method shown in Section 3.1 is 1.17 seconds (which includes plotting the result shown in Figure 1) using MATLAB on a 700MHz Pentium III PC. For comparison, the algorithm proposed in [11] takes 18 seconds using the same parameters.

### 4.2 Norm-bounded nonlinear systems

We consider a norm-bounded nonlinear system

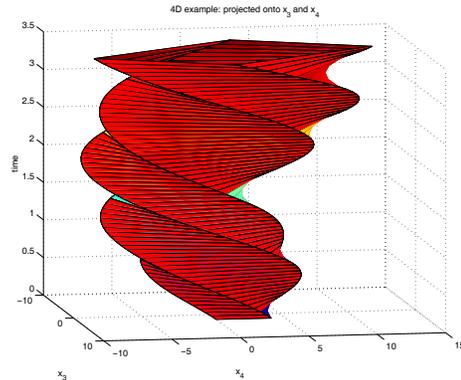
$$\dot{x} = A(t)x + B(t)u(t) + \phi(x, t), \quad x(0) \in X_0, \quad u(t) \in U \quad (33)$$

where the initial set  $X_0$  and the control input set  $U$  are convex polytopes. The nonlinear function  $\phi(x, t)$  is assumed to be norm-bounded i.e.,  $\|\phi(x, t)\| \leq \frac{1}{3}t$  where  $t > 0$ . The system parameters are defined as follows:

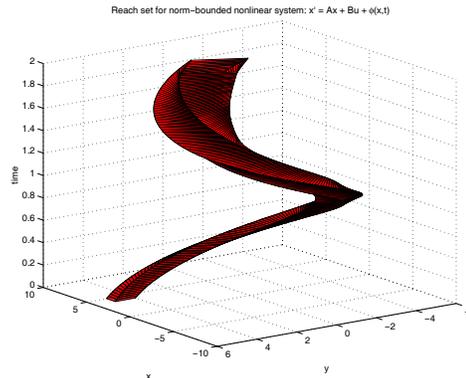
$$A = \begin{bmatrix} -0.5 & 4.0 \\ -3.0 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$X_0 = [4, 5] \times [4, 5], \quad U = [-0.1, 0.1]$$

The evolution of the forward reachable set over time is shown in Figure 2 and its computation time is 0.87 seconds (including plotting the result) using MATLAB on the same PC.



**Figure 1:** The forward reachable set of a four dimensional linear dynamical system (projection onto  $x_3$  and  $x_4$ ).



**Figure 2:** The forward reachable set of a norm-bounded nonlinear system.

### 4.3 Conflict resolution between two aircraft

Last, we consider the two aircraft collision avoidance problem, as an example of feedback linearizable nonlinear systems and linear dynamic games. This is the same problem (the motivation for this research) described in Section 2. Aircraft 1 tries to avoid a conflict with aircraft 2 no matter how aircraft 2 behaves, within the limits of its capability. Thus, we want to compute a backward reachable set (unsafe set) from the target set (protected zone) which represents the states from which the two aircraft would eventually have a conflict no matter how aircraft 1 tries to avoid it [5].

Using dynamic extension [22] with  $\sigma_i$  as a new state variable (compared to (5)), we obtain a new nonlinear model which is feedback linearizable [23],

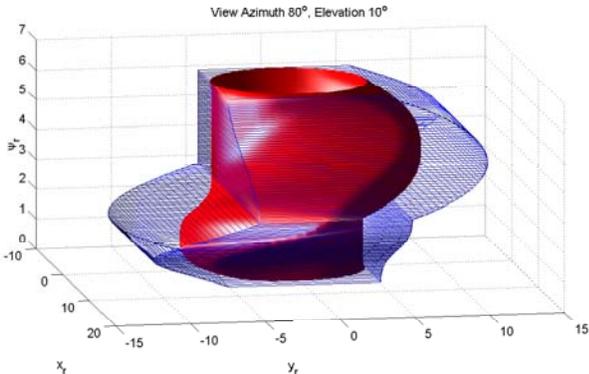
$$\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\psi}_i \\ \dot{\sigma}_i \end{bmatrix} = \begin{bmatrix} \sigma_i \cos \psi_i \\ \sigma_i \sin \psi_i \\ \omega_i \\ a_i \end{bmatrix}, \quad (i \in \{1, 2\}) \quad (34)$$

where  $a_i$  is the acceleration of aircraft  $i$  and is a new control input. Thus, the new state and input variables are  $\xi_i := [x_i \ y_i \ \psi_i \ \sigma_i]^T$  and  $\eta_i := [a_i \ \omega_i]^T$  respectively. We introduce a change of the state variables  $z_i = T(\xi_i)$  and a change of the input variables  $\eta_i = M(\xi_i)u_i$  as in [23]. We denote that  $T$  and  $M$  are diffeomorphisms

everywhere except at  $\sigma_i = 0$ . Then, the feedback linearized model of the nonlinear kinematic aircraft model in (34) obtained through the transformations  $T$  and  $M$  is [23]:

$$\dot{z}_i = \frac{\partial T}{\partial \xi_i} \dot{\xi}_i \Rightarrow \dot{z}_i = Az_i + Bu_i \quad (35)$$

with  $A$  and  $B$  defined in [23]. The relative kinematic

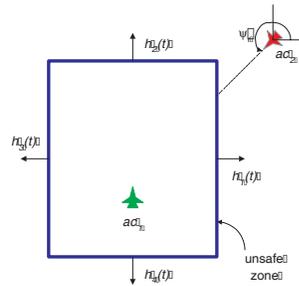


**Figure 3:** Comparison between overapproximate (grid) and exact (solid) backward reachable sets (unsafe sets) of conflict resolution between two aircraft.

aircraft model between two aircraft can be obtained by introducing new states  $\xi_r := \xi_2 - \xi_1$  in the original nonlinear state space and  $z_r := z_2 - z_1$  in the linearized state space. Thus, a linearized relative kinematic aircraft model is

$$\dot{z}_r = Az_r + Bu_2 - Bu_1, \quad u_2 \in U, u_1 \in D, \quad (36)$$

where the admissible control input set  $U$  and the disturbance input set  $D$  are polytopes. This is a linear dynamic game since aircraft 1 ( $u_1$ ) tries to keep aircraft 2 from entering into its protected zone (target set) to prevent a conflict, but aircraft 2 ( $u_2$ ) tries to enter the protected zone of aircraft 1. A target set (protected zone) is assumed to be  $\mathcal{Y}_0 = [-5, 5] \times [-5, 5] \times [-\pi, \pi]$ . Using dynamic extension, we have performed the computation in four dimensions (36) and projected the result onto the relative coordinate in three-dimensional space. A polytopic overapproximate backward reachable set is first computed in the linearized space and then the overapproximate backward reachable set in the original state space is obtained through the transformations  $T$  and  $M$ . The overapproximate backward reachable set for conflict resolution with heading changes only, using the target set  $\mathcal{Y}_0$ , normalized aircraft speeds  $\sigma_1 = \sigma_2 = 5$ , angular velocities  $|\omega_1| \leq 1$  and  $|\omega_2| \leq 1$  is compared with the exact solution in [4] in Figure 3. The backward reachable set obtained by using the polytopic approximation is overapproximate of the exact reachable set and its computation time is about 1.0 seconds (including plotting the result as shown in Figure 3) using MATLAB on the same PC, yet the exact solution [5] takes approximately 15 minutes on a Sun

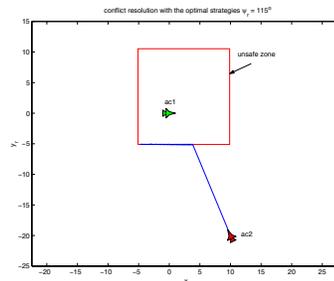


**Figure 4:** Conflict scenario: Aircraft 2 reaches the boundary of the unsafe zone of aircraft 1 with a given initial relative angle  $\psi_r$ .

UltraSparc II with 50 grid nodes in each dimension. Figure 4 shows a conflict scenario in which aircraft 2 tries to enter the unsafe zone. When aircraft 2 reaches the boundary of the unsafe zone as shown in Figure 4, the optimal control input for aircraft 1 can be easily obtained as follows:

$$\begin{aligned} u_1^*(t) &= \arg \max_{u_1 \in D} \{ \langle D_x v(x, t), -B(t)u_1(t) \rangle \} \\ &= \arg \max_j \langle e^{-A^T t} h_1(0), -Bu_1^j \rangle \end{aligned} \quad (37)$$

Figure 5 shows a simulation for conflict resolution



**Figure 5:** Conflict resolution simulation with relative initial states ( $x_r = 10, y_r = -20, \psi_r = 115^\circ$ ). Aircraft 1 tries to avoid a conflict with aircraft 2 with the optimal strategy.

between the two aircraft with the initial condition ( $x_r = 10, y_r = -20, \psi_r = 115^\circ$ ). Since both aircraft behave optimally, the relative position of aircraft 2 moves along the boundary of the unsafe set. As expected, chattering occurs along the boundary. To avoid such a phenomenon, we could introduce a buffer zone around the boundary so that the control inputs change smoothly as aircraft 2 approaches the boundary.

## 5 Conclusions

The polytopic approximation gives an overapproximation of the exact reachable set and is computationally efficient: it requires solving matrix exponentials instead of a Hamilton-Jacobi partial differential equation. The data structure of the polytopic approximation method becomes more complicated than that of the ellipsoidal approximation method [15] as the number of faces of the polytope increases, yet the compu-

tation of the matrix exponential is easier than solving the (usually Riccati type) differential equation required for the ellipsoidal methods. The optimal control input can be easily computed from the Hamiltonian since the Hamiltonian is linear with respect to the control, and the control input set is a convex polytope. The polytopic approximation method can be applied to high dimensional systems which may not be solved exactly without substantially increasing the computational time.

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