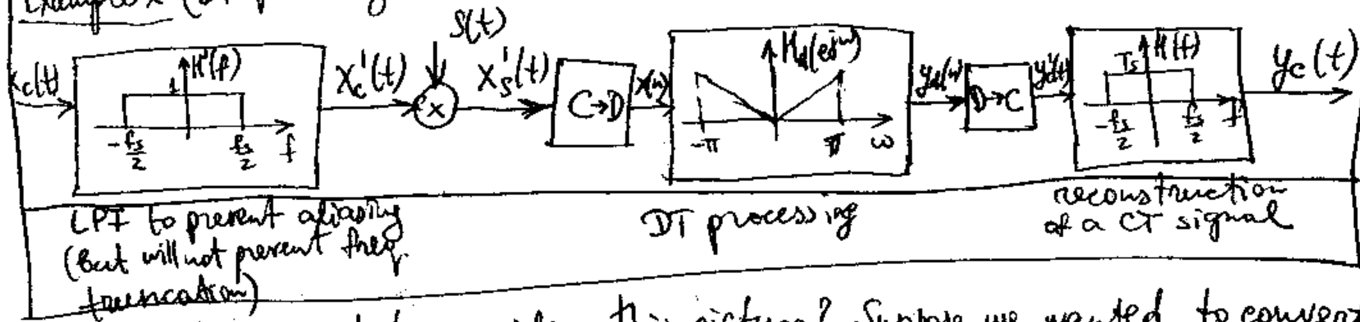
1.4 Sampling (continued).

Example 2 (DT processing of CT signals)



Why would we want to consider this picture? Suppose we wanted to convert an old analog recording to digital format, store it on a compact disc, and then play it back on a CD player.
This picture illustrates - in a very idealized manner - the steps we would take. Note that, before storing the signal on a CD, we might want to do some signal processing, for example, in order to enhance the quality of the audio signal.
(Of course, this is a toy example: you wouldn't use this particular filter $H_d(e^{j\omega})$ in reality; but the structure of this process is quite similar.)
Once you got your CD, you want to play it using your audio equipment - that is, you want to convert the discrete-time signal on the CD into a continuous-time music signal.

Last time: considered everything but the middle portion of the diagram.
Let us now look at what's happening in this picture, for a very simple input.

Suppose $X_c(f) = \frac{1}{2}\left(\delta(f-a)+\delta(f+a)\right)$

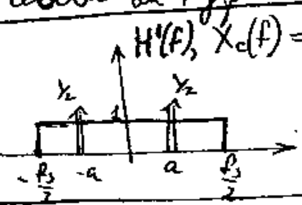
What's $x_c(t)$?

$$\boxed{\begin{aligned} x_c(t) &= \int_{-\infty}^{\infty} X_c(f) e^{j2\pi f t}\, df \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \delta(f-a) e^{j2\pi f t}\, df + \frac{1}{2}\int_{-\infty}^{\infty} \delta(f+a) e^{j2\pi f t}\, df \\ &= \frac{1}{2}e^{j2\pi a t} + \frac{1}{2}e^{-j2\pi a t} \\ &= \cos(2\pi a t) \end{aligned}}$$

(exercise: use inverse CTFT)

Case 1: Sampling frequency is above the Nyquist rate, $f_s > 2a$.

1. $X_c'(f) = X_c(f)\,H'(f)$

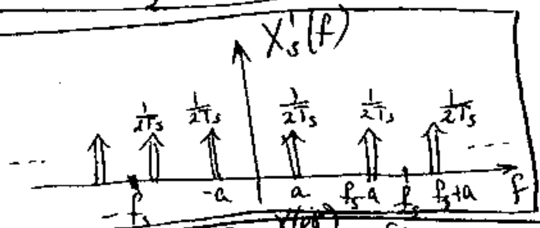


2. $X_s'(f) = X_c'(f) * S(f)$

$$= X_c'(f) * \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_s}\right)$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X_c'\left(f-\frac{n}{T_s}\right)$$

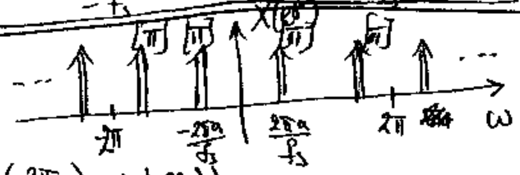
(Recall: $f_s = \frac{1}{T_s}$)



3. $X(e^{j\omega}) = X_s'\left(\frac{\omega}{2\pi T_s}\right)$

rescale the freq. axis :

(e.g. $X(e^{j\omega})|_{\omega=2\pi} = X_s'\left(\frac{2\pi}{2\pi T_s}\right) = X_s'(f_s)$)



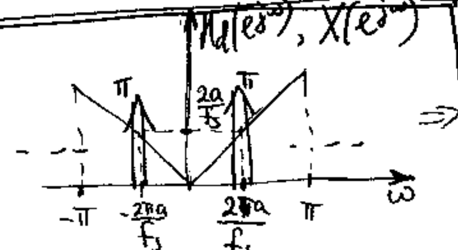
Is there anything else we must do?

Adjust the areas of δ's: recall that $\delta(a\omega) = \frac{1}{|a|}\delta(\omega)$, and so

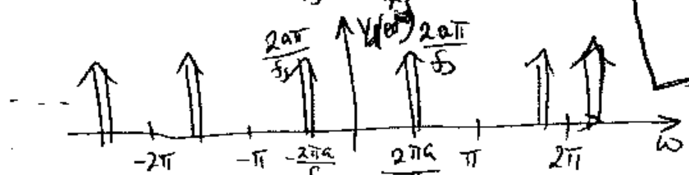
$$\frac{1}{2T_s}\,\delta\left(\frac{\omega}{2\pi T_s}\right) = \frac{2\pi T_s}{2T_s}\,\delta(\omega) = \pi\,\delta(\omega)$$

4. $Y_d(e^{j\omega}) = X(e^{j\omega})\,H_d(e^{j\omega})$

$$H_d(e^{j\omega}) = \begin{cases} \frac{\omega}{\pi}, & 0 \le \omega \le \pi \\ -\frac{\omega}{\pi}, & -\pi \le \omega \le 0 \end{cases}$$



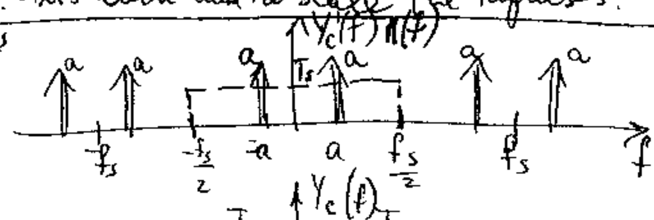
$\Rightarrow$



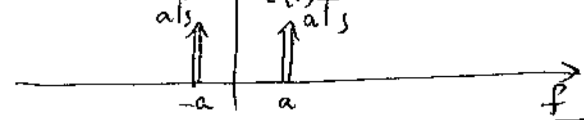
5. To get $Y_c'(f)$, re-label the freq. axis back and re-scale the impulses.

6. $Y_c(f) = Y_c'(f)\,H(f) = \begin{cases} T_s Y_c'(f), & |f| \le \frac{f_s}{2} \\ 0, & |f| > \frac{f_s}{2} \end{cases}$

$$= T_s a\big(\delta(f-a) + \delta(f+a)\big)$$



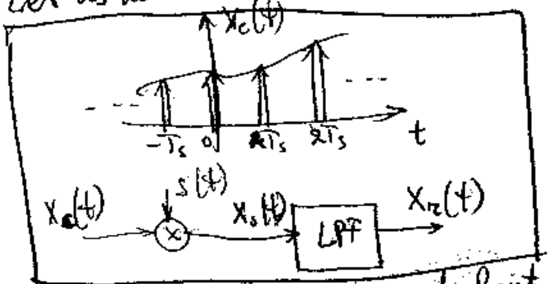
$y(t) = 2aT_s\cos(2\pi a t)$ $\Leftarrow$



~~What will~~ Case 2: $f_s < 2a$. What will happen?

Answer: zero, because everything will be filtered out by the first LPF.

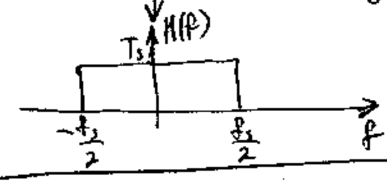
Let us now look a little closer at the reconstruction:



[Forget for the moment about pre-filtering]

$$x_s(t) = \sum_n x_c(nT_s)\,\delta(t-nT_s)$$

reconstruct by LPF



Reconstructed spectrum:

$$X_r(f) = X_s(f)H(f) \Rightarrow$$

$$x_r(t) = x_s(t) * \underbrace{h(t)}_{\parallel}$$

$$= x_s(t) * \mathrm{sinc}(f_s t) = \left\{\sum_n x_c(nT_s)\,\delta(t-nT_s)\right\} * \mathrm{sinc}(f_s t)$$

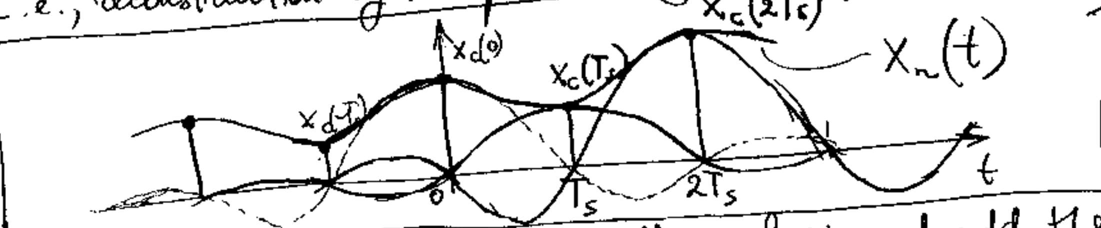
$$= \sum_n x_c(nT_s)\,\mathrm{sinc}(f_s(t-nT_s))$$

ICFT: $\displaystyle\int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,df =$

$$= T_s \frac{1}{j2\pi t} e^{j2\pi ft}\Big|_{f=-\frac{f_s}{2}}^{f=\frac{f_s}{2}}$$

$$= \mathrm{sinc}(f_s t)$$

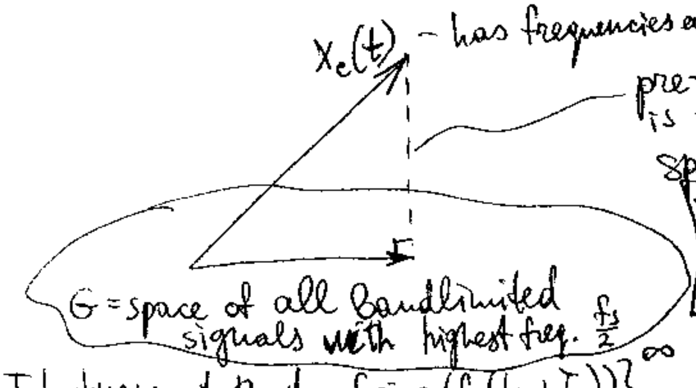
I.e., reconstruction by low-pass filtering is interpolation with sinc functions



LEFT Board DNE

Scale the sincs by the corresponding values, and add them all together

Again, $x_r(t) = \sum_k a_k g_k(t)$



– has frequencies above $\frac{f_s}{2}$

pre-filtering with an LPF to avoid aliasing is the orthogonal projection onto this space.

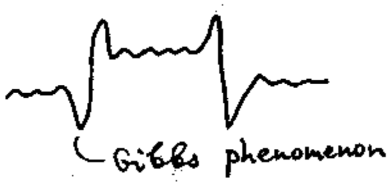
This results in a reconstruction which is the closest to $x_c(t)$ among all possible signals in G.

It turns out that $\{\mathrm{sinc}(f_s(t-nT_s))\}_{n=-\infty}^{\infty}$ = orth. basis for this space

Why is this interpretation important? Well, the space of all bandlimited signals is good for approximating smooth signals whose Fourier transform has energy concentrated at low frequency. It's also well adapted to sound recordings, which are well approximated by lower frequency harmonics.

For discontinuous signals, such as images, a low-frequency restriction produces the Gibbs oscillations which you have seen in Lab 3. If you try to approximate something like this:

with a few low-frequency components, you get something like this:



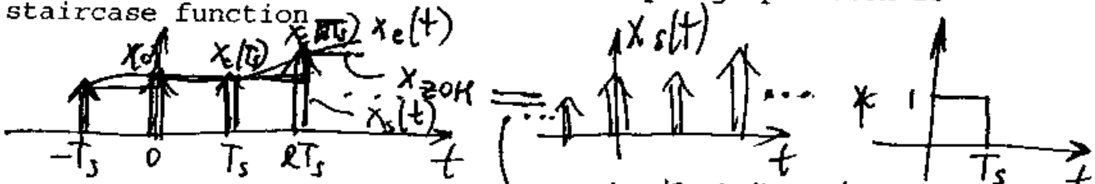
The visual quality of images is degraded by these oscillations, so, it is not a good idea to sample images in the same way you sample sound signals. So, for the sampling and reconstruction of images, you may want to pick a basis which is different from the sinc basis, and project onto a different subspace. But the basic paradigm will remain the same.

And, again, this is yet another reason why our exercises in linear algebra were very useful. In signal processing, it is very important to get used to thinking about signals as vectors. Then you start seeing the forest behind the trees: namely, that much of what we do here is just decomposing signals into different bases, and working with projections of signals onto the bases vectors. We've seen that this is the basic idea behind convolution, frequency analysis, and now sampling.

In Lab 4, you consider several deviations from the ideal situation that we've looked at so far.

Effects of Zero-Order Hold Sampling.

First of all, it is impossible to produce ideal impulses of infinite energy and zero duration. What you are considering instead in Lab 4 is a sample-and-hold scheme, when you sample a value and hold it until the next sample. So, instead of an impulse train, the result of this sampling operation is a staircase function



Note that the staircase function can simply be written as the convolution of the impulse train $x_s(t)$ with a box fn

(5)

$$\boxed{X_{ZOH}(f) = X_s(f)\cdot \text{CTFT}\left\{ \text{rect}_{[0,T_s]} \right\}} \text{ - exercise}$$

$$\int_0^{T_s} e^{-j2\pi f t}\,dt = \frac{1}{\pi f}\left[-\frac{1}{2j}e^{-j2\pi f t}\Big|_0^{T_s}\right]$$

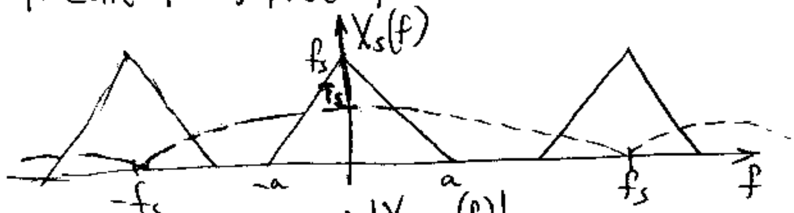
$$= \frac{1}{\pi f}\left[\frac{1}{2j}\left(1-e^{-j2\pi f T_s}\right)\right]$$

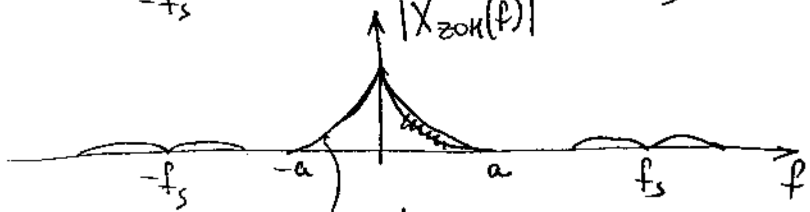
$$= \frac{1}{\pi f}e^{-j\pi f T_s}\left[\frac{1}{2j}\left(e^{j\pi f T_s}-e^{-j\pi f T_s}\right)\right]$$

$$= T_s e^{-j\pi f T_s}\frac{\sin(\pi f T_s)}{\pi f T_s} = T_s e^{-j\pi f T_s}\,\text{sinc}(f T_s)$$

$$= X_s(f)\,T_s e^{-j\pi f T_s}\,\text{sinc}(f T_s)$$

So, $|X_{ZOH}(f)| = T_s\cdot|X_s(f)|\cdot|\text{sinc}(fT_s)|$





Possible solution: oversample, $f_s \gg a$