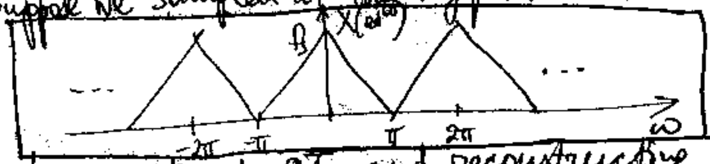
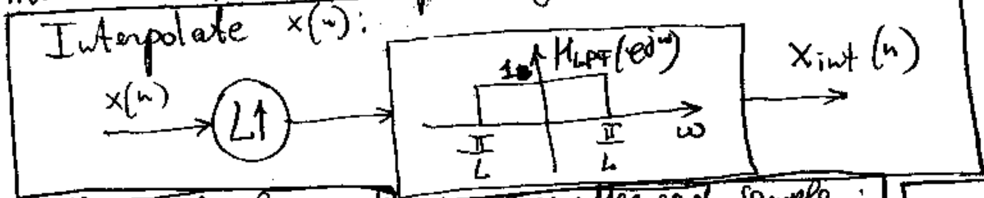
DT Interpolation (Increasing the Sampling Rate).

Another important deviation from the ideal scenario is that it is impossible to build an ideal analog low-pass filter — that is, a filter, which would be exactly a non-zero constant for some range of frequencies, and exactly zero everywhere. Moreover, it is very difficult and expensive to build even a good approximation to such a filter. ~~What are the implications?~~ It is much easier to build such digital filter. What are the implications?

Suppose we sampled at the Nyquist rate, barely avoiding aliasing:



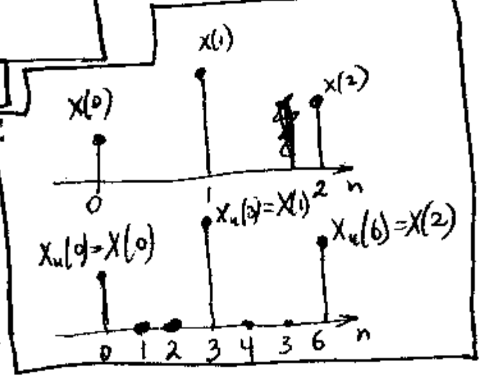
Just converting to CT and reconstructing will not work, as we would need an ideal filter. Instead, we could partially solve this problem in DT:



(a) Upsample by inserting $L-1$ zeros after each sample:

$$\begin{cases} x_u(Ln) = x(n) \\ x_u(Ln+1) = x_u(Ln+2) = \ldots = x_u(Ln+L-1) = 0 \end{cases}$$

E.g., $L=3$:

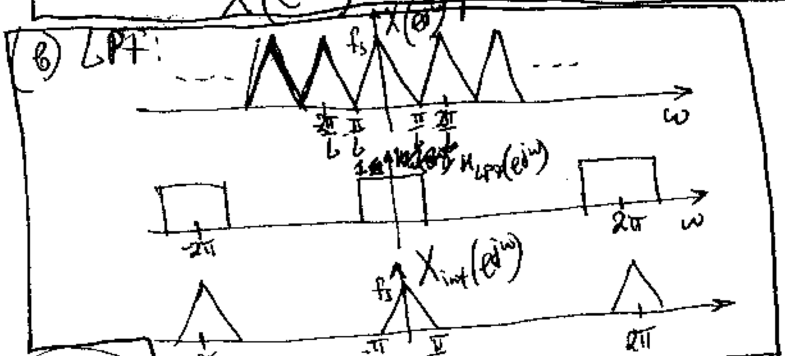


$$X_u(e^{j\omega}) = \sum_k x_u(k)e^{-j\omega k} = \sum_n x_u(Ln)e^{-j\omega Ln} = \sum_n x(n)e^{-j(\omega L)n}$$

$\neq 0$ only if k is an integer multiple of L

$= X(e^{j\omega L})$ — Eq. (6) in Lab 4.

(b) LPF:



Now we can get away with a poor analog LPF, and still reconstruct the original signal very well.

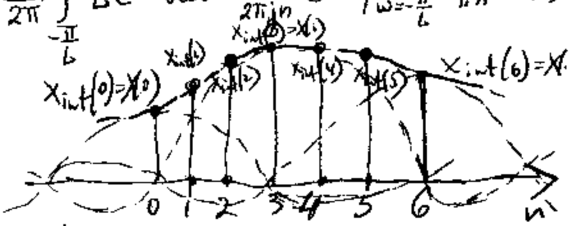
Imp. resp. of the LPF

$$\boxed{x_{int}(n) = x_u * h(n)} = \sum_k x_u(k)\,\mathrm{sinc}\left(\frac{n-k}{L}\right) = \sum_m x_u(mL)\,\mathrm{sinc}\left(\frac{n-mL}{L}\right) = \sum_m x(m)\,\mathrm{sinc}\left(\frac{n-mL}{L}\right)$$

$$\boxed{\begin{aligned} h(n) &= \mathrm{IDTFT}\{H_{LPF}\} \\ &= \mathrm{sinc}\left(\frac{n}{L}\right) \end{aligned}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LPF}(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} L e^{j\omega n}\,d\omega = \frac{L}{2\pi jn} e^{j\omega n}\Big|_{\omega=-\frac{\pi}{L}}^{\omega=\frac{\pi}{L}} = \frac{L}{\pi n}\sin\left(\frac{\pi n}{L}\right)$$

Similar to CT reconstruction:

$$\boxed{x_r(t) = \sum_n x(n)\,\mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right)}$$

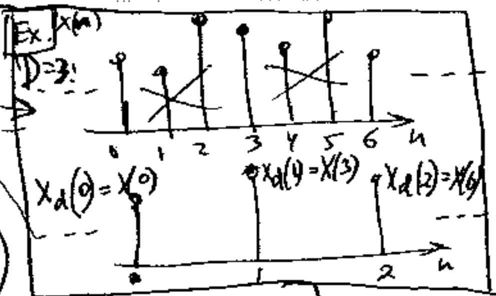


Good for slowly varying signals; not so good for signals with sharp transitions, because of Gibbs oscillations.

③

Decimation (Decreasing Sampling Rate).

Downsampling: $x_d(n) = x(Dn)$
(take every D-th sample)



Recall: if $x(n) = \underbrace{x_c(nT_s)}_{\text{CT signal}}$,

then $$X(e^{j\omega}) = \frac{1}{T_s}\sum_n X_c\left(\frac{\omega - 2\pi n}{2\pi T_s}\right)$$

Since $x_d(n) = x_c(nDT_s)$, replace $T_s \to DT_s$ above:

$$X_d(e^{j\omega}) = \frac{1}{DT_s}\sum_n X_c\left(\frac{\omega-2\pi n}{2\pi DT_s}\right)$$

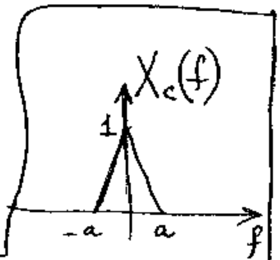
Relate $X_d(e^{j\omega})$ to $X(e^{j\omega})$:

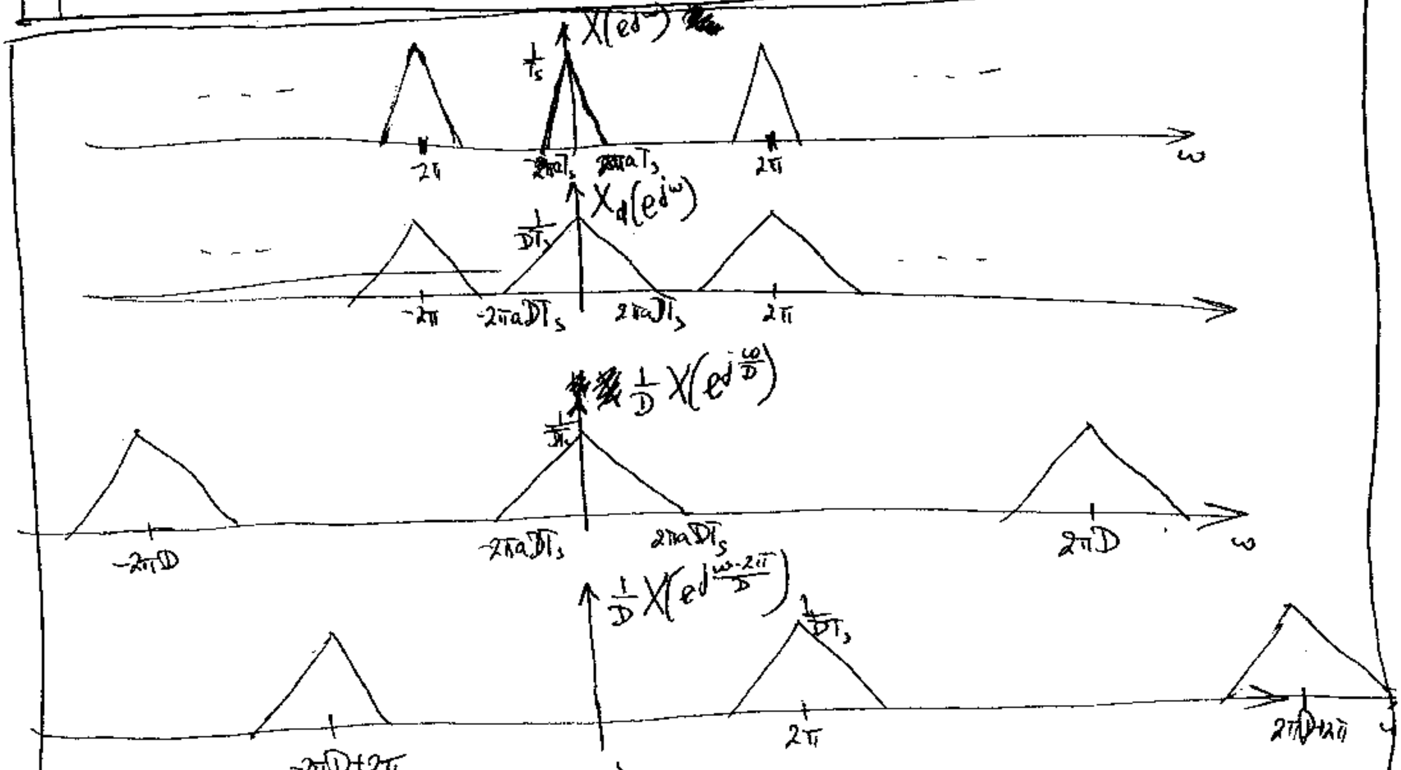
Change of variables $n = k + rD$ with $-\infty < r < \infty$, $0 \le k \le D-1$

$$X_d(e^{j\omega}) = \frac{1}{D}\sum_{k=0}^{D-1}\left[\frac{1}{T_s}\sum_{r=-\infty}^{\infty} X_c\left(\frac{(\omega-2\pi k)-2\pi r D}{2\pi DT_s}\right)\right]$$

$$= \frac{1}{D}\sum_{k=0}^{D-1}\left[\frac{1}{T_s}\sum_{r=-\infty}^{\infty} X_c\left(\frac{\frac{\omega-2\pi k}{D}-2\pi r}{2\pi T_s}\right)\right]$$

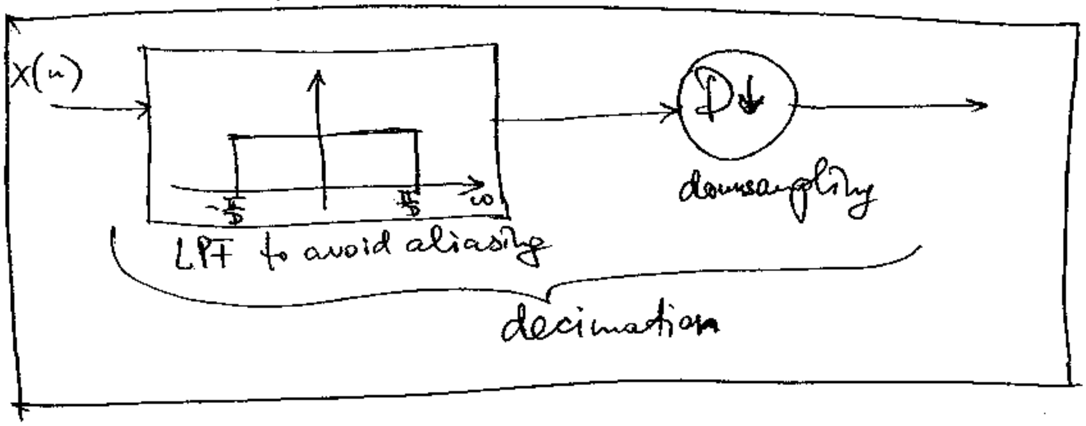
$$= \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j\frac{\omega-2\pi k}{D}}\right).$$





$2\pi a T_s D > \pi$ i.e., $2\pi a T_s > \frac{\pi}{D}$ $\Rightarrow$ aliasing

Just as in CT, pre-filter to avoid aliasing:

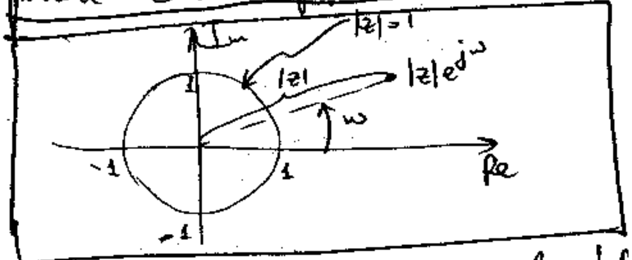


1.5 Z-Transform.

An important tool for filter design, and for analysing stability of systems.

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n},}$$

where z is complex: $z = |z|\, e^{j\omega}$



DTFT is the Z-transform, evaluated on the unit circle:

$$\boxed{X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}}}$$

1.5.1 Rational Z-Transforms.

We will mostly be interested in Z-transforms which are rational function of z = ratio of two polynomials

$$X(z) = \frac{P(z)}{Q(z)} \quad \text{polynomials in } z$$

Consider a linear, constant coefficient difference eq.:

$$y(n) = \sum_{i=0}^{N-1} b_i\, x(n-i) - \sum_{k=1}^{M} a_k\, y(n-k)$$

(Terminology:
- if no 2nd term, the system in <u>non-recursive</u>, or <u>finite-duration impulse response (FIR)</u>
- otherwise, <u>recursive</u> (current y is expressed in terms of past y's)
  if cannot be written as non-recursive, then <u>infinite-duration impulse response (IIR)</u>.)

$$ZT\{x(n-i)\} = z^{-i} X(z) \quad \text{- exercise}$$

$$Y(z) = \sum_{i=0}^{N-1} b_i z^{-i} X(z) - \sum_{k=1}^{M} a_k z^{-k} Y(z)$$

$$Y(z)\left(1 + \sum_{k=1}^{M} a_k z^{-k}\right) = X(z) \sum_{i=0}^{N-1} b_i z^{-i}$$

Lab 5 Wk 1 pg 3

assume $b_0 \neq 0$

Transfer fcn: $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{\sum_{i=0}^{N-1} b_i z^{-i}}{1 + \sum_{k=1}^{M} a_k z^{-k}} = \dfrac{z^{-N+1} \sum_{i=0}^{N-1} b_i z^{N-1-i}}{z^{-M}\left(z^M + \sum_{k=1}^{M} a_k z^{M-k}\right)} = b_0 z^{M-N+1} \dfrac{\prod_{i=1}^{N-1} (z - z_i)}{\prod_{k=1}^{M} (z - p_k)}$

(since polynomials of degrees $N-1$ and M have $N-1$ and M roots, respectively)