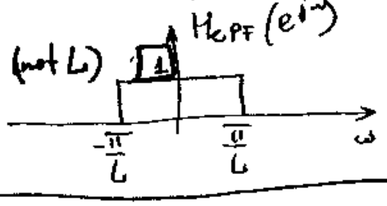
Correction for interpolation LPF:



1.5 Z-transform.

Last time, we started studying the Z-transform, which, as you are seeing in this week's lab, is an important tool, in particular, for filter design.

1.5.1. Rational Z-Transforms.

$$y(n) = \sum_{i=0}^{N-1} b_i x(n-i) - \sum_{k=1}^{M} a_k y(n-k)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N-1} b_i z^{-i}}{1+\sum_{k=1}^{M} a_k z^{-k}} \underset{\text{if } b_0 \neq 0}{=} b_0 z^{M-N+1} \frac{\prod_{i=1}^{N-1}\overbrace{(z-z_i)}^{\text{zeros}}}{\prod_{k=1}^{M}\underbrace{(z-p_k)}_{\text{poles}}}$$

Values of z for which $X(z)=0$ are called the *zeros* of $X(z)$:

$z_1, \dots, z_{N-1}$ – roots of the numerator

Values of z for which $X(z)$ is infinite are called the *poles* of $X(z)$:

$p_1, \dots, p_M$ – roots of the denominator

In addition, $M-N+1$ zeros at $z=0$ if $M>N-1$, or

$N-1-M$ poles at $z=0$ if $N-1>M$

Poles and zeros may also occur at $z=\infty$:

"zero at ∞" means $\lim_{|z|\to\infty} X(z)=0$

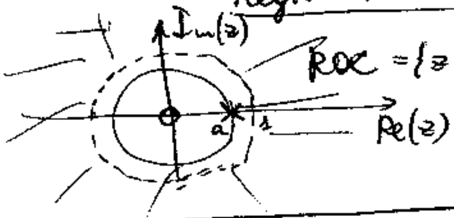
"pole at ∞" means $\lim_{|z|\to\infty} X(z)=\infty$.

Example. $ZT\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$ if $|az^{-1}|<1$

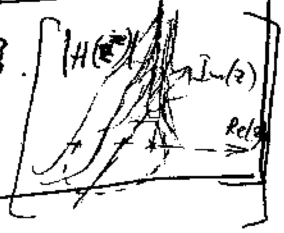
region of convergence (ROC): $|z|>|a|$

Pole: $z=a$

Zero: $z=0$



ROC $=\{z: |z|>|a|\}$.



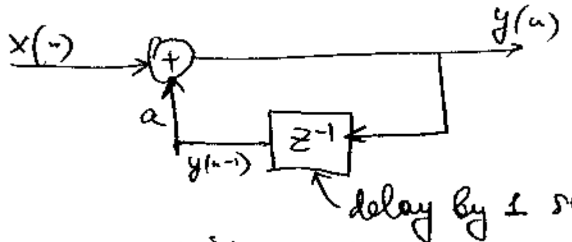
Let's analyze the filter whose transfer fcn is this

Let $H(z)=\frac{z}{z-a}$

(a) Diff. eq.:

$y(n) = ay(n-1)+x(n)$ Recursive IIR

(b) Diagram:

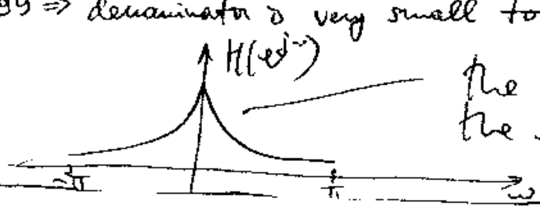


(c) Freq. resp: $H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$

$$|H(e^{j\omega})| = \left|\frac{1}{1-ae^{-j\omega}}\right| = \frac{1}{\sqrt{(1-ae^{-j\omega})(1-ae^{j\omega})}} = \frac{1}{\sqrt{1-2a\cos\omega+a^2}}$$

$$= \frac{1}{\sqrt{(1-a)^2+2a(1-\cos\omega)}}$$

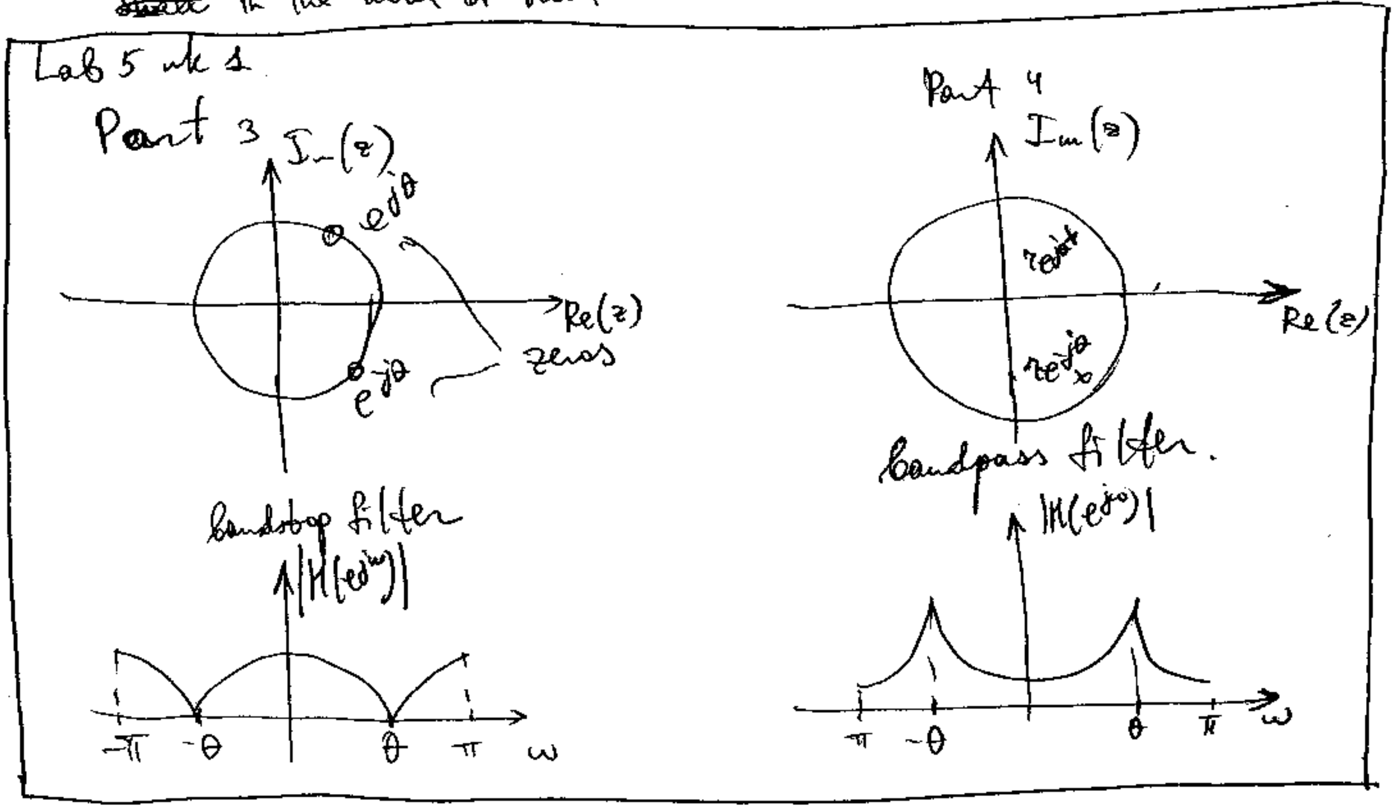
$a=0.99 \Rightarrow$ denominator is very small for $\omega=0$:



The closer "a" to the unit circle, the sharper the peak, the thinner the passband

In general,

- a pole near the unit circle will cause the freq. resp. to ~~be~~ increase ~~large~~ in the nbhd of that pole
- a zero near the unit circle will cause the freq. resp. to ~~be~~ decrease ~~small~~ in the nbhd of that zero.



1.5.2 Region of Convergence (ROC) of the Z-Transform.

Example 1. $\boxed{x(n) = 2^n u(n)}$

$$\boxed{X(z) = \sum_{n=0}^{\infty} 2^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n}$$

When can we sum this series? (I.e., when does this series converge?)

(a) If $|z| \le 2$, then $\left|\frac{2}{z}\right| \ge 1$

$\Rightarrow$ every term has abs. value ≥ 1 and growing (or ~~non-in~~ constant)

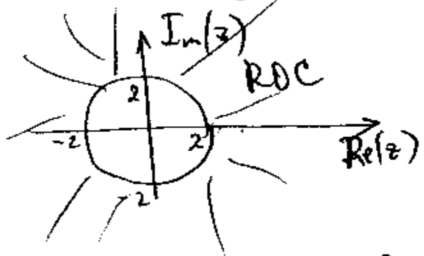
$\Rightarrow$ can't sum

(b) If $|z| > 2$, then $\left|\frac{2}{z}\right| < 1$

$$\Rightarrow \boxed{X(z) = \frac{1}{1-\frac{2}{z}}}$$

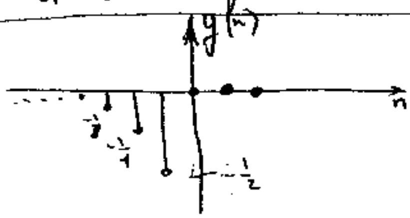
Thus, the Z-transform of $x(n) = 2^n u(n)$ is

$$X(z) = \begin{cases} \frac{1}{1-\frac{2}{z}}, & |z| > 2 \\ \text{undefined}, & |z| \le 2 \end{cases}$$



Usually called "the region of convergence (ROC) of the Z-transform", because it's where this series actually converges to this function. A slightly better term would be "the region of definition", since outside this region, the Z-transform of this sequence is simply undefined.

Example 2. $\boxed{y(n) = -2^n u(-n-1)}$



$$Y(z) = \sum_{n=-1}^{-\infty} -2^n z^{-n} = -\sum_{m=0}^{\infty} 2^{-m-1} z^{m+1} =$$

$m = -n-1$
$n = -m-1$

$$= -\frac{z}{2} \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^m$$

(a) $|z| \ge 2 \Rightarrow$ can't sum

(b) $|z| < 2 \Rightarrow \boxed{Y(z) = -\frac{z}{2} \frac{1}{1-\frac{z}{2}} = -\frac{z}{2-z} = \frac{1}{1-\frac{2}{z}}}$

$$Y(z) = \begin{cases} \text{undefined}, & |z| \ge 2 \\ \frac{1}{1-\frac{2}{z}}, & |z| < 2 \end{cases}$$

The same expression for the Z-transform as in Example 1, but non-intersecting regions of convergence.

Example 3. $w(n) = 2^n, \quad -\infty < n < \infty.$

$$w(n) = 2^n u(n) + 2^n u(-n-1) = x(n) - y(n)$$

$$W(z) = X(z) - Y(z) = \frac{1}{1-\frac{2}{z}} - \frac{1}{1-\frac{2}{z}} = 0 \quad ?$$

What's wrong with this?

$X(z)$ and $Y(z)$ have no common ROC:

$X(z)$ is undefined for $|z| \leq 2$

$Y(z)$ " " $|z| \leq 2$

$\Rightarrow W(z)$ is never defined.

Def. ROC of $\sum_{n=-\infty}^{\infty} x(n) z^{-n}$ is the set of all z for which this series is absolutely convergent:

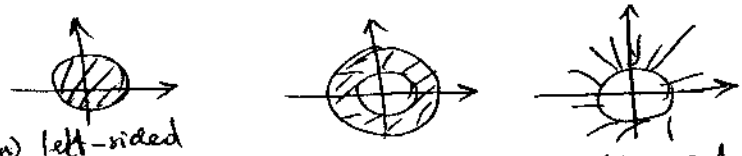
$$\sum_{n=-\infty}^{\infty} |x(n) z^{-n}| < \infty$$

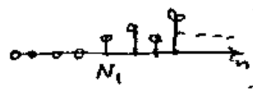
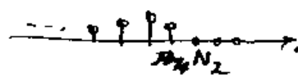
(We use absolute convergence to avoid certain pathological series which converge, but do not converge absolutely.)

~~Properties~~

1.5.3. Properties of ROC, Poles, and Zeros.

1) The ROC is a ring or a disc centered at the origin:
$r_1 < |z| < r_2$ (r_1 and/or r_2 could be 0 or ∞)



2) The ROC cannot contain any poles.
3) If $x(n)$ is a finite-duration sequence (i.e., $x(n)=0$ except for $N_1 \le n \le N_2$), then ROC is the whole z-plane except possibly $z=0$.
4) If $x(n)$ is a right-sided sequence ($x(n)=0$ for $n<N_1$ )
then ROC is $|z| > |\underbrace{p_{max}}_{\text{outermost finite pole of } X(z)}|$
5) If $x(n)$ is left-sided ($x(n)=0$ for $n > N_2$ )
then ROC is $|z| < |\underbrace{p_{min}}_{\text{innermost non-zero pole in } X(z)}|$
6) An LTI system is BIBO stable if and only if the ROC of its transfer fcn $H(z)$ includes $z=1$ (i.e., includes the unit circle):

$$\left[\sum_n |h(n) z^{-n}|\right]_{z=1} < \infty$$

$$\Updownarrow$$

$$\sum_n |h(n)| < \infty$$

$$\Updownarrow$$

BIBO stable

(Again, <u>ONLY</u> if LTI).