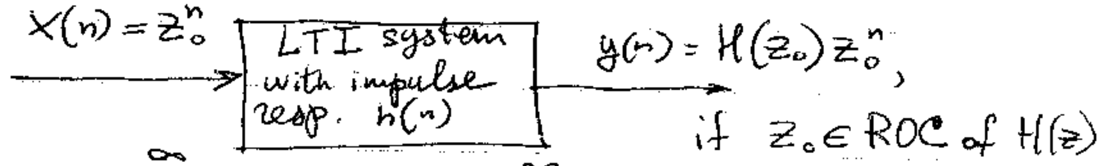
1.5.4. DT exponentials z_0^n are eigenfunctions of DT LTI systems.



$$y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k) = \sum_{k=-\infty}^{\infty} h(k) z_0^{n-k} = z_0^n \sum_{k=-\infty}^{\infty} h(k) z_0^{-k}$$

$$= z_0^n \, ZT\{h(k)\}\Big|_{z=z_0} = z_0^n H(z_0)$$

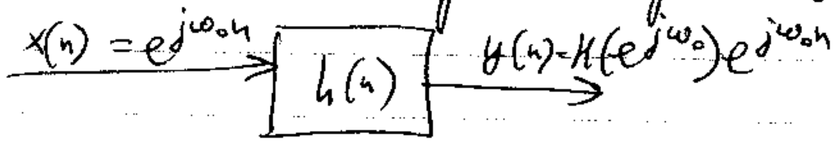
if z_0 is in the ROC

Transfer function $H(z)$ is:

- the z-transform of the impulse response $h(n)$
- the scaling factor of z^n when z^n goes through the system.

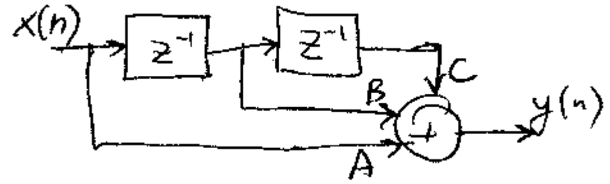
Recall that we have already considered the case of $z_0 = e^{j\omega_0}$:

Frequency response $H(e^{j\omega_0})$ is:

- the DTFT of the impulse response $h(n)$
- 

Exam

Example: HW 4 Prob 4



(a) Difference equation:

$$y(n) = A\,x(n) + B\,x(n-1) + C\,x(n-2)$$

(b)(iii) Find $H(e^{j\omega})$ by calculating the response to $e^{j\omega n}$.

$$x(n) = e^{j\omega n} \Rightarrow y(n) = A e^{j\omega n} + B e^{j\omega(n-1)} + C e^{j\omega(n-2)}$$

$$= \underbrace{\left(A + Be^{-j\omega} + Ce^{-2j\omega}\right)}_{H(e^{j\omega})} e^{j\omega n}$$

(c)(ii) Suppose $A = B = C = 1$ and $x(n) = 5$. $y(n) = ?$

$$x(n) = 5e^{j\cdot 0 n} \Rightarrow y(n) = H(e^{j0})\left(5e^{j\cdot 0\cdot n}\right)$$

$$= (1+1+1)\,5 = 15, \text{ for all } n.$$

Example: HW 4 Prob 2.

(a) (i)(iii) If $x(n) = \cos(\omega n + \varphi)$, is it always true that $y(n) = \cos(\omega n + \varphi)H(e^{j\omega})$?

$$\cos(\omega n + \varphi) = \tfrac{1}{2}\left(e^{j(\omega n + \varphi)} + e^{-j(\omega n+\varphi)}\right)$$

$$= \tfrac{1}{2} e^{j\varphi} e^{j\omega n} + \tfrac{1}{2} e^{-j\varphi} e^{-j\omega n}$$

$$y(n) = \tfrac{1}{2} e^{j\varphi} H(e^{j\omega}) e^{j\omega n} + \tfrac{1}{2} e^{-j\varphi} H(e^{-j\omega}) e^{-j\omega n}$$

Unless $H(e^{j\omega}) = H(e^{-j\omega})$, this is not equal to $\cos(\omega n+\varphi)H(e^{j\omega})$.

NOTE: in general, $\cos(\omega n + \varphi)$, $e^{j\omega n}u(n)$, $z^n u(n)$, are NOT eigenfunctions of LTI systems.

(ii) If $x(n) = 2\cos\left(8n+\frac{\pi}{4}\right)$,

$$y(n) = 2\left\{\frac{1}{2}e^{j\frac{\pi}{4}}H(e^{8j})e^{j8n} + \frac{1}{2}e^{-j\frac{\pi}{4}}H(e^{-8j})e^{-j8n}\right\}$$

$$= H(e^{8j})e^{j(8n+\frac{\pi}{4})} + H(e^{-8j})e^{-j(8n+\frac{\pi}{4})}.$$

(iv) Suppose $H(e^{j\omega}) = H(e^{-j\omega})$, and

$$H(e^{j\omega}) = \frac{1+ae^{-j2\omega}+2e^{-j4\omega}}{1+be^{-j2\omega}}$$

Determine a, b, given:

- the resp. to $\cos\frac{\pi n}{2}$ is $\cos\frac{\pi n}{2}$
- the resp. to $\sin(\pi n)$ is $2\sin(\pi n)$

$$\begin{cases} H(e^{j\frac{\pi}{2}}) = 1 \\ H(e^{j\pi}) = 2 \end{cases} \Rightarrow \begin{cases} \dfrac{1+ae^{-j\pi}+2e^{-j2\pi}}{1+be^{-j\pi}} = 1 \\ \dfrac{1+ae^{-j2\pi}+2e^{-j4\pi}}{1+be^{-j2\pi}} = 2 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{1-a+2}{1-b} = 1 \\ \dfrac{1+a+2}{1+b} = 2 \end{cases} \Rightarrow \begin{cases} 3-a = 1-b \\ 3+a = 2+2b \end{cases}$$

$$\Rightarrow \begin{cases} b = a-2 \\ 3+a = 2+2a-4 \end{cases} \Rightarrow \begin{cases} b=3 \\ a=5. \end{cases}$$

(6) (i) Response to $e^{j\frac{\pi}{2}n}$ is $4e^{j\left(\frac{3\pi}{4}\right)n}$.

Can the system satisfy $y(n)=(x*h)(n)$?

NO: different frequencies.

(ii) $e^{j\frac{\pi}{4}n} \rightarrow 4e^{j\frac{9\pi}{4}(n-1)} = 4e^{-j\frac{\pi}{4}}e^{j\frac{\pi}{4}n}$ YES, e.g. $H(e^{j\omega})=4e^{-j\frac{\pi}{4}}$.

Several remarks about HW1–HW4.

~~Bounded~~

Finite <u>number</u>: e.g., 5 is finite, while $\sum_{n=0}^{\infty} n$ is not.

Bounded <u>signal</u> $x(n)$: if there is a constant M such that $|x(n)| \leq M$, <u>for all</u> n. E.g., $x(n)=n$, $-\infty < n < \infty$, is an <u>unbounded</u> signal: even though each value of $x(n)$ is finite, there is no constant M which bounds <u>all</u> of them at once.

~~Summable~~

~~Finite-duration~~

Absolutely summable <u>signal</u> $x(n)$: if $\sum_{n=-\infty}^{\infty} |x(n)|$ is a finite number.

Absolutely summable $\Rightarrow$ bounded

bounded $\not\Rightarrow$ absolutely summable

e.g., $x(n)=1$ for all n is bounded, but $\sum_{n=-\infty}^{\infty} x(n) = \infty$.

Example: HW2 Prob 3.

Suppose $x(n)$ is bounded and $h(n)$ is absolutely summable. Can $y(n) = x*h(n)$ be unbounded?

No.

Wrong proof: $h(n)$ is absolutely summable $\Rightarrow h(n)$ is bounded.

Bounded $*$ bounded = bounded ✗

If $x(n)=h(n)=1$ for all n, then $x*h(n)$ is <u>not</u> bounded (in fact, is not defined).

Correct proof (done in class):

$$|y(n)| = \left|\sum_{k=-\infty}^{\infty} x(k)\,h(n-k)\right|$$

$$\leq \sum_{k=-\infty}^{\infty} |x(k)|\cdot|h(n-k)|$$

$$\leq \sum_{k=-\infty}^{\infty} M\cdot|h(n-k)| =$$

if each $|x(k)|$ is less than some constant M

$$= M \underbrace{\sum_{k=-\infty}^{\infty} |h(n-k)|}_{\text{finite number } L \text{, since } h \text{ is absolutely summable}} = ML$$

$|y(n)| \leq ML$ for all $n \Rightarrow y$ is a bounded signal

~~Problem~~ Example: HW 2 Prob 2(c).

Is $y(n) = \begin{cases} x(n), & \text{if } x(n) \leq 1 \\ 1, & \text{otherwise} \end{cases}$ stable?

~~Wrong~~ Answer: yes.

Wrong proof #1. Let $x(n)=1$. Then $y(n)=1$. Bounded input led to a bounded output $\Rightarrow$ system is BIBO stable.

ONE EXAMPLE IS NOT ENOUGH: NEED TO SHOW THIS FOR EVERY input-output pair

Wrong proof #2. Impulse response $h(n)=\delta(n)$, is absolutely summable $\Rightarrow$ the system is BIBO stable

Correct proof. If $|x(n)| \leq M$ for all n,

$|y(n)| \leq \max(1,M)$ for all n.