Lec 15 9/26/01 Wed

Exam1:

Problem 1c. Energy of $x(n) = \begin{cases} \cos(\pi n), & 1 \le n \le 2001 \\ 0, & \text{else} \end{cases}$

Sol'n. $\sum_{n=-\infty}^{\infty} \underbrace{|\cos(\pi n)|^2}_{1} = \underbrace{1+1+1+\dots+1}_{2001 \text{ times}} = \underline{2001}.$

Problem 3c. Is $y(n) = (n+1)x(-n^2)$ causal?

Sol'n. Can n be $< -n^2$? $n + n^2 < 0 \Rightarrow -1 < n < 0$ — but this cannot happen since n is integer! The system is causal.

Note: the continuous-time system

$$y(t) = (t+1)\,x(-t^2), \quad -\infty < t < \infty,$$

is NOT causal.

Problem 4. An LTI system with $H(e^{j\cdot 0}) = H(e^{j\frac{4\pi}{3}}) = 0$, $H(e^{j\frac{2\pi}{3}}) = 1$.

(a) Response to $\underbrace{\ldots, 1, -\frac{1}{2}+\frac{\sqrt{3}}{2}j, -\frac{1}{2}-\frac{\sqrt{3}}{2}j, 1, -\frac{1}{2}+\frac{\sqrt{3}}{2}j, -\frac{1}{2}-\frac{\sqrt{3}}{2}j, \ldots}_{e^{j\frac{2\pi}{3}n}}$

Use:

$$e^{j\omega_0 n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow H(e^{j\omega_0})\, e^{j\omega_0 n}$$

Here, $\omega_0 = \frac{2\pi}{3}$, $H(e^{j\omega_0}) = H(e^{j\frac{2\pi}{3}}) = 1$,
so the response to $e^{j\frac{2\pi}{3}n}$ is $e^{j\frac{2\pi}{3}n}$.

(b) Response to an arbitrary 3-periodic signal $s(n)$.

- represent $s(n)$ as a Fourier series
- use Part (a) for each of the frequency components

$$s(n) = a_0 e^{j\frac{2\pi\cdot 0}{3}n} + a_1 e^{j\frac{2\pi\cdot 1}{3}n} + a_2 e^{j\frac{2\pi\cdot 2}{3}n}$$

$$= a_0 + a_1 e^{j\frac{2\pi}{3}n} + a_2 e^{j\frac{4\pi}{3}n}$$

$\Rightarrow$ response to $s(n)$ is

$$y(n) = a_0 \underbrace{H(e^{j\cdot 0})}_{0} + a_1 \underbrace{H(e^{j\frac{2\pi}{3}})}_{1} e^{j\frac{2\pi}{3}n} + a_2 \underbrace{H(e^{j\frac{4\pi}{3}})}_{0} e^{j\frac{4\pi}{3}n}$$

$$= a_1 e^{j\frac{2\pi}{3}n}$$

$$y(438) = y(0) = a_1$$

$$a_1 \underset{\uparrow}{=} \frac{1}{3}\left\langle s(n), e^{j\frac{2\pi\cdot 1}{3}n}\right\rangle = \frac{1}{3}\left(s(0) + s(1)e^{-j\frac{2\pi}{3}} + s(2)e^{-j\frac{4\pi}{3}}\right)$$

Fourier series formula

1.6 Discrete Fourier Transform (DFT) and Fast Fourier Transform.

1.6.1 Introduction and Definitions.

Our whole motivation for studying frequency analysis can be summarized in the following picture:

$$x(n) = \sum_{k=0}^{N-1} X(k)\, g_k(n) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y(n) = \sum_{k=0}^{N-1} X(k)\, H\!\left(e^{j\frac{2\pi k}{N}}\right) g_k(n)$$

$x(n)$: N-periodic (or duration N)

$$g_k(n) = \frac{1}{N} e^{j\frac{2\pi k}{N} n}, \quad k = 0, \dots, N-1; \text{ all } n.$$

- conceptual importance:
  LTI systems "process" each harmonic separately and in a very simple way (i.e., multiplying it by a frequency-dependent complex number).
- computational importance:
  - to obtain $X(k) H\left(e^{j\frac{2\pi k}{N}}\right)$ from $X(k)$ for $k = 0, \dots, N-1$, need only N operations
  - to obtain $X(k)$ from $x(n)$ and $y(n)$ from $X(k) H\left(e^{j\frac{2\pi k}{N}}\right)$, need only $O(N \log N)$ operations. (FFT)

This picture, to a large extent, is a summary of this course.

- At the beginning of the course, we looked at the basic ideas and terminology for the components of this picture – i.e., signals and systems.
- We then started studying this picture directly in Topic 1.3, Frequency Analysis.
- We then looked at two very important practical issues:
  - Sampling – because most real-world signals are continuous-space and continuous-time. We discussed how CT signals can be reliably converted into DT signals and back.
  - Filter design – to underscore the importance of thinking about signals and systems in the frequency domain. Along the way, we introduced an important tool for systems analysis and for solving linear difference equations, namely, z-transform.

We are now going back to frequency analysis, to look at fast algorithms for computing Fourier series coefficients. (4)

In vector form,

$$\underline{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix}, \quad \boxed{g_k = \begin{pmatrix} \frac{1}{N} e^{j\frac{2\pi k}{N}\cdot 0} \\ \frac{1}{N} e^{j\frac{2\pi k}{N}\cdot 1} \\ \vdots \\ \frac{1}{N} e^{j\frac{2\pi k}{N}(N-1)} \end{pmatrix}} \quad \text{for } k=0,1,\dots,N-1. \qquad (*)$$

Then $\underline{x} = \sum_{k=0}^{N-1} X(k) g_k$.

Since g_k's are pairwise orthogonal, we can use the projection formula to calculate the coefficients:

$$X(k) = \frac{\langle \underline{x}, g_k\rangle}{\langle g_k, g_k\rangle} = \frac{\sum_{n=0}^{N-1} x(n)\, g_k^*(n)}{\left(\frac{1}{N}\right)} = N \sum_{n=0}^{N-1} x(n) \frac{1}{N} e^{-j\frac{2\pi k}{N}n} =$$

from HW

$$= \boxed{\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N}n}}, \quad k=0,\dots,N-1$$

DFT

Def. The DFT of an N-periodic (or N-point) signal $x(n)$ is the sequence ~~of~~ $X(k)$ of its Fourier series coefficients in the basis $(*)$:

$$x(n) = \sum_{k=0}^{N-1} X(k)\, g_k(n) = \boxed{\frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi k}{N}n}}, \quad n=0,\dots,N-1$$

IDFT

Remark 1. DFT $\neq$ DTFT (although related)

DFT: discrete "frequency" k

DTFT: cont. frequency ω

Remark 2 $e^{j\frac{2\pi k}{N}n}$ is periodic as a fcn of n, with period N

$\Rightarrow$ IDFT defines a periodic signal for $-\infty < n < \infty$

$\Rightarrow$ we will often think of $x(n)$ as N-periodic.

Also, $e^{-j\frac{2\pi k}{N}n}$ is N-periodic as a fcn of $\underline{\underline{k}}$.

$\Rightarrow$ DFT defines a periodic signal $X(k)$ for $-\infty < k < \infty$

$\Rightarrow$ we will often think of $X(k)$ as N-periodic.