1.6 DFT & FFT

1.6.1 Introduction (continued).

$$g_k(n) = \frac{1}{N} e^{j\frac{2\pi k}{N} n}, \quad k = 0, 1, \ldots, N-1$$

Last time, we reviewed DT Fourier series for this particular basis of complex exponentials, and called the resulting sequence of Fourier series coefficients $X(k)$ the Discrete Fourier Transform (DFT) of $x(n)$. (This is a standard term used in literature.)

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N} n}, \quad \text{and} \quad x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi k}{N} n}}$$

In vector form,

$$\underline{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix}, \quad g_k = \begin{pmatrix} \frac{1}{N} e^{j\frac{2\pi k}{N}\cdot 0} \\ \frac{1}{N} e^{j\frac{2\pi k}{N}\cdot 1} \\ \vdots \\ \frac{1}{N} e^{j\frac{2\pi k}{N}(N-1)} \end{pmatrix}$$

Then $\underline{x} = \sum_{k=0}^{N-1} X(k) g_k$

Reminder: Lab 6-1 (Next week), part 3.2.

Let $\underline{X} = \begin{pmatrix} X(0) \\ \vdots \\ X(N-1) \end{pmatrix}$, then

$$\underline{x} = \underbrace{\left(g_0 \;\; g_1 \;\; g_2 \;\; \cdots \;\; g_{N-1}\right)}_{B \,=\, N\times N \text{ matrix whose columns are } g_k\text{'s}} \underline{X} = B\underline{X} \qquad \text{IDFT}$$

(because $\left(g_0 \;\; g_1 \;\; \cdots \;\; g_{N-1}\right) \begin{pmatrix} X(0) \\ \vdots \\ X(N-1) \end{pmatrix} = X(0)g_0 + X(1)g_1 + \cdots + X(N-1)g_{N-1}$

$$= \sum_{k=0}^{N-1} X(k) g_k)$$

Entries of this matrix B are:

$$B_{nk} = \frac{1}{N} e^{j\frac{2\pi (k-1)(n-1)}{N}}$$

n-th row, $n = 1, 2, \ldots, N$; k-th column, $k = 1, 2, \ldots, N$

To get the DFT formula, premultiply $\underline{x} = B\underline{X}$ by the matrix $A = NB^H$,
where $y^H = (y^*)^T$ means "conjugate transpose of y"

$$A = N \underbrace{\begin{pmatrix} g_0^H \\ g_1^H \\ \vdots \\ g_{N-1}^H \end{pmatrix}}_{N\times N \text{ matrix whose rows are conjugate transposes of } g_k\text{'s}}$$

$$g_k^H = \left(\frac{1}{N} e^{-j\frac{2\pi k}{N}\cdot 0} \;\; \frac{1}{N} e^{-j\frac{2\pi k}{N}\cdot 1} \;\; \cdots \;\; \frac{1}{N} e^{-j\frac{2\pi k}{N}(N-1)}\right).$$

$$A_{kn} = NB^*_{nk} = e^{-j\frac{2\pi (k-1)(n-1)}{N}}$$ Eq. (16) from Lab 6 wk 1

k-th row, $k = 1, 2, \ldots, N$; n-th column, $n = 1, 2, \ldots, N$

$A\underline{x} = AB\underline{X}$.

$AB = ?$

$$N\begin{pmatrix} g_0^H \\ g_1^H \\ \vdots \\ g_{N-1}^H \end{pmatrix}\begin{pmatrix} g_0 & g_1 & \cdots & g_{N-1}\end{pmatrix} = N\begin{pmatrix} g_0^H g_0 & g_0^H g_1 & \cdots & g_0^H g_{N-1} \\ g_1^H g_0 & g_1^H g_1 & \cdots & g_1^H g_{N-1} \\ \vdots & & & \\ g_{N-1}^H g_0 & g_{N-1}^H g_1 & \cdots & g_{N-1}^H g_{N-1}\end{pmatrix}$$

(Note: $g_k^H g_p = \langle g_p, g_k \rangle = \sum_{n=0}^{N-1} g_p(n)\, g_k^*(n)$)

$$= N\begin{pmatrix} \frac{1}{N} & & 0 \\ & \frac{1}{N} & \\ 0 & & \ddots \;\frac{1}{N}\end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \;1\end{pmatrix} = I$$

(identity matrix),

i.e., A and B are inverses of each other.

$$\boxed{\underline{X} = A\underline{x}}$$

DFT

$$\underset{N\times 1}{\underline{X}} = \underset{N\times N}{A}\;\underset{N\times 1}{\underline{x}}$$

Need N^2 complex multiplications for a brute-force implementation of DFT.

(Lab 6-1 3.3: variations on this ___bad___ implementation.)

However, the fact that the matrix A is highly structured can be exploited to produce a much faster algorithm for multiplying a vector by this matrix A.

1.6.2 FFT (Lab 6-2 Section 3).

Fast Fourier Transform (FFT) is any algorithm of computational complexity $O(N\log N)$ for computing the N-point DFT.

It is NOT a new transform.

Assume N is an integer power of 2

$$\underbrace{X^{(N)}(k)}_{\substack{N\text{-point DFT}\\ \text{of } x(n)}} = \sum_{n \text{ is even: } n=2m,\ m=0,1,\dots,\frac{N}{2}-1} x(n)e^{-j\frac{2\pi k}{N}n} + \sum_{n \text{ is odd: } n=2\ell+1,\ \ell=0,1,\dots,\frac{N}{2}-1} x(n)e^{-j\frac{2\pi k}{N}n}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x(2m)e^{-j\frac{2\pi k}{N}2m} + \sum_{\ell=0}^{\frac{N}{2}-1} x(2\ell+1)e^{-j\frac{2\pi k}{N}(2\ell+1)}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x_0(m)e^{-j\frac{2\pi k}{(\frac{N}{2})}m} + e^{-j\frac{2\pi k}{N}}\sum_{\ell=0}^{\frac{N}{2}-1} x_1(\ell)e^{-j\frac{2\pi k}{(\frac{N}{2})}\ell}$$

$$= X_0^{(\frac{N}{2})}(k) + e^{-j\frac{2\pi k}{N}} X_1^{(\frac{N}{2})}(k) \qquad (*)$$

(Let $x_0(m) = x(2m)$, $m=0,\dots,\frac{N}{2}-1$
$x_1(\ell) = x(2\ell+1)$, $\ell=0,\dots,\frac{N}{2}-1$)

$\frac{N}{2}$-point DFT of even-numbered samples of $x(n)$ ($\frac{N}{2}$-periodic)

$\frac{N}{2}$-point DFT of odd-numbered samples of $x(n)$ ($\frac{N}{2}$-periodic)

To compute $X^{(N)}(k)$ for $k=0,\dots,N-1$,

- compute $X_0^{(\frac{N}{2})}(k)$ for $k=0,\dots,\frac{N}{2}-1$
- compute $X_1^{(\frac{N}{2})}(k)$ for $k=0,\dots,\frac{N}{2}-1$
- then do $(*)$ with N complex multiplications and N complex additions [actually, ~~slightly~~ fewer]

Let $W_N = e^{-j\frac{2\pi}{N}}$

$$W_N^{k+\frac{N}{2}} = e^{-j\left(\frac{2\pi k}{N}+\pi\right)} = -e^{-j\frac{2\pi k}{N}} = -W_N^k$$

So,

$$X^{(N)}(k) = X_0^{(\frac{N}{2})}(k) + W_N^k X_1(k) \quad \text{for } k=0 \dots \tfrac{N}{2}-1 \qquad \text{Eq. (12) Lab 6-2.}$$

$$X^{(N)}\left(k+\tfrac{N}{2}\right) = X_0^{(\frac{N}{2})}(k) - W_N^k X_1(k) \quad \text{"}$$