1.6.3. Fast Computation of Convolution. Lec 18, 10/3/01 Wed

Consider a linear system described by

$$\underline{y} = S\,\underline{x}$$

$\underline{y}$: $N\times 1$ output vector, representing an N-periodic output signal.

S: $N\times N$ matrix

$\underline{x}$: $N\times 1$ input vector, representing an N-periodic input signal

What does S have to satisfy in order for the system to be TI, i.e. invariant to (circular) shifts?

$$\begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix} \xrightarrow{\text{circular shift by 1 sample.}} \begin{pmatrix} x(-1)=x(N-1) \\ x(0) \\ x(1) \\ \vdots \\ x(N-2) \end{pmatrix}$$

Let the first column of S be $\underline{h} = \begin{pmatrix} h(0) \\ h(1) \\ \vdots \\ h(N-1) \end{pmatrix}$.

Note that when $\underline{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\underline{y} = \underline{h}$

when $\underline{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\underline{y}$ = 2nd column of S which must therefore be $\begin{pmatrix} h(N-1) \\ h(0) \\ h(1) \\ \vdots \\ h(N-2) \end{pmatrix}$.

when $\underline{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\underline{y}$ = 3rd column of S, etc.

$$S = \begin{pmatrix} h(0) & h(N-1) & h(N-2) & \cdots & h(1) \\ h(1) & h(0) & h(N-1) & \cdots & h(2) \\ h(2) & h(1) & h(0) & \cdots & h(3) \\ \vdots & \vdots & \vdots & & \vdots \\ h(N-1) & h(N-2) & h(N-3) & \cdots & h(0) \end{pmatrix}$$ – a <u>circulant</u> matrix

$$y(n) = \sum_{m=0}^{N-1} x(m)\, h(n-m)$$

$$= \sum_{m=0}^{N-1} x(m)\, h((n-m) \bmod N)$$ ~ this formula works even when x and/or h are non-periodic

$$= x \,Ⓝ\, h(n) = x \circledast h(n)$$ – circular (or periodic) convolution.

(E.g., $y(0) = x(0)h(0) + x(1)h(N-1) + x(2)h(N-2) + \dots + x(N-1)h(1)$ –
the sum of the indexes of x and h is always 0 mod N

$y(1) = x(0)h(1) + x(1)h(0) + x(2)h(N-1) + \dots + x(N-1)h(2)$
the sum is always 1 mod N, etc.)

What are the eigenvectors of S? Try $\frac{1}{N} e^{j\frac{2\pi k}{N}n}$:

$$g_k = \begin{pmatrix} \frac{1}{N} e^{j\frac{2\pi k}{N}\cdot 1} \\ \frac{1}{N} e^{j\frac{2\pi k}{N}\cdot 2} \\ \vdots \\ \frac{1}{N} e^{j\frac{2\pi k}{N}(N-1)} \end{pmatrix}, \quad k = 0, 1, \dots, N-1.$$

$$y(n) = \sum_{m=0}^{N-1} h(m)\, g_k(n-m) = \sum_{m=0}^{N-1} h(m) \frac{1}{N} e^{j\frac{2\pi k}{N}(n-m)}$$

$$= \left\{ \sum_{m=0}^{N-1} h(m) e^{-j\frac{2\pi k}{N}m} \right\} \frac{1}{N} e^{j\frac{2\pi k}{N}n}$$

$$= \underbrace{H(k)}_{\text{DFT of } h} \frac{1}{N} e^{j\frac{2\pi k}{N}n}$$

$$S g_k = H(k)\, g_k$$

g_k ← kth eigenvectors of S

$H(k)$ ← corresponding eigenvalue

$$S \cdot \underbrace{(g_0 \; g_1 \; \cdots \; g_{N-1})}_{\text{the IDFT matrix } B} = (g_0 \; g_1 \; \cdots \; g_{N-1}) \begin{pmatrix} H(0) & & & 0 \\ & H(1) & & \\ & & \ddots & \\ 0 & & & H(N-1) \end{pmatrix}$$

$$S = B \begin{pmatrix} H(0) & & & 0 \\ & H(1) & & \\ & & \ddots & \\ 0 & & & H(N-1) \end{pmatrix} A,$$

where DFT matrix A is:

$$A = NB^H = \begin{pmatrix} g_0^H \\ g_1^H \\ \vdots \\ g_{N-1}^H \end{pmatrix}$$

Complex exponentials are <u>eigenvectors</u> of circulant matrices. They <u>diagonalize</u> circulant matrices.

Thus, for any $\underline{x} \in \mathbb{C}^N$,

$$S\underline{x} = B \begin{pmatrix} H(0) & & 0 \\ & \ddots & \\ 0 & & H(N-1) \end{pmatrix} A\underline{x}$$

Two algorithms for computing the circular convolution of $\underline{x}$ and $\underline{h}$:

<u>Alg. 1.</u> Perform multiplication $S\underline{x}$ $\quad O(N^2)$.

<u>Alg. 2.</u> <u>Step 1.</u> Represent $\underline{x}$ in the eigenbasis of S — the Fourier basis

$$\underline{X} = A\underline{x} \qquad O(N \log N) \text{ with FFT}$$

<u>Step 2.</u> Compute the representation of y in the eigenbasis of S:

$$\underline{Y} = \begin{pmatrix} H(0) & & 0 \\ & \ddots & \\ 0 & & H(N-1) \end{pmatrix} \underline{X} \qquad O(N)$$

<u>Step 3.</u> Reconstruct y from its Fourier coefficients:

$$\underline{y} = B\underline{Y} \qquad O(N \log N) \text{ with FFT}$$

Total complexity: $O(N \log N)$

(Note: Alg. 2 is not necessarily better for <u>any</u> matrix.)

Step 1	Step 2	Step 3
$x(n) \xrightarrow{\text{N-pt DFT}} X(k)$ $h(n) \xrightarrow{\text{N-pt DFT}} H(k)$	$Y(k) = X(k)H(k)$	$Y(k) \xrightarrow{\text{N-pt IDFT}} y(n) = x \circledast h(n)$.

[Note: this development is essentially the solution to HW8 Prob 1.]

1.6.3.1. The relationship between convolution and periodic convolution.

④

Let x, h be N-periodic signals, and let

$$x_z = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$h_z = \begin{cases} h(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Let $y_z(n) = x_z * h_z(n)$,

$y(n) = x \circledN h(n)$.

Then

$$y(n) = \begin{cases} y_z(n) + y_z(N+n), & n = 0, 1, \cdots, N-2 \\ y(N-1), & n = N-1 \end{cases}$$ — temporal aliasing