## 1.6.3.1. The relationship between convolution and periodic convolution (continued).

(Lec 19), Fri 10/5/01 (1)

Let $x, h$ be $N$-periodic signals, and let

$$x_z = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$h_z = \begin{cases} h(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

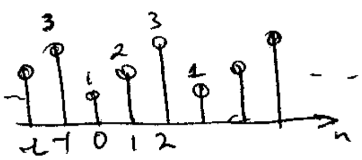
Let $y_z(n) = x_z * h_z(n)$,

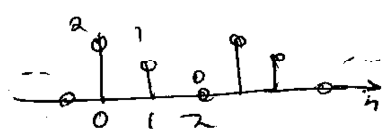
$y(n) = x \circledast_N h(n)$.

Then

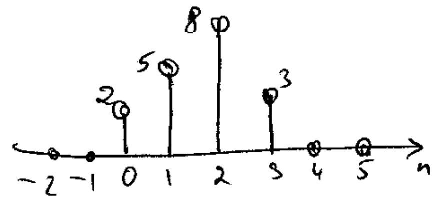
$$y(n) = \begin{cases} y_z(n) + y_z(N+n), & n = 0, 1, \dots, N-2 \\ y(N-1), & n = N-1 \end{cases}$$ — temporal aliasing

(HW 8)

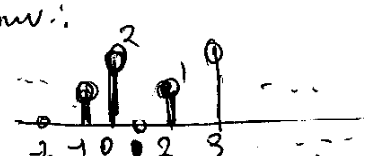
Example: $x(n) =$ 

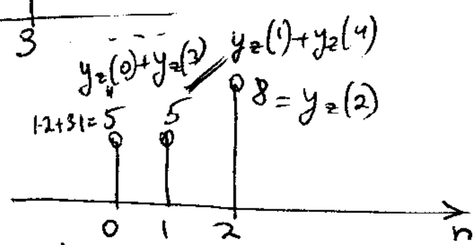
, $h(n) =$ 


(a) Convolution:

$y_z(n) = x_z * h_z(n) =$ 


(b) Circular conv.:

$h(-m) =$ 


$y(n) = x \circledast_3 h(n) =$ 


How can we calculate the aperiodic convolution using FFT?

HW 8

## 1.7 Random Signals.

So far in this course, we only considered deterministic signals. In many practical situations, this is not enough. It's much more convenient to model many processes as random signals.

For example, if you take images with a medical imaging device, there will always be noise in the picture, and the exact intensity values will vary from picture to picture. They will depend on a lot of different factors which are beyond our control, and which therefore cannot be reliably modeled. However, what we may be able to reliably model is the average behavior of these signals. In other words, if we do some averaging procedure over a hundred images, perhaps we will be able to predict how the next hundred images will behave on average. Once we have a good probabilistic model, we could ask and answer a lot of useful questions about estimating the images, to remove noise and enhance the quality.

So, our objective for this topic will be to develop the analysis tools for random signals, and to ultimately be able to process random signals.

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                          --------
random signal x(n)       |  LTI   |     random signal y(n)
----------------------->  |        |  --------------------->
                          | system |
                          --------
```

(Note. Probabilistic modelling—i.e., assigning probabilities to events—will not be studied in this course. We'll always assume that either a prob. model is given, or it can be easily constructed)

We'll start at the very beginning, by reviewing some basic facts about probability from EE 302.

### 1.7.1. Events and Probability.

Main concepts:

- "Outcome of an experiment", or an "elementary event".

- Sample space $\Omega$ = collection of all possible outcomes.

- Event = a set in the sample space (for us, ANY set).
  We say that an event A occurs if one of the outcomes in the set A occurs.

- Probability measure is a function which ~~assigns a probability~~ (a number) P(A) to each event A, ~~and which~~ satisfies the following ~~properties:~~

  1. It is non-negative.
  2. The probability of the whole sample space is 1.
  3. It is countably additive.

1. $P(A) \geq 0$ for any event $A$
2. $P(\Omega) = 1$
3. $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ ~~for~~ if $A_i \cap A_j = \phi$ for any $i, j$

Recall:

$(A_1 + A_2)$ $A_1 \cup A_2$ = union of $A_1, A_2$ = "$A_1$ or $A_2$"

$(A_1 A_2) = A_1 \cap A_2$ = intersection of $A_1, A_2$ = "$A_1$ and $A_2$" [$A_1 \cap A_2 = 0$ means mutually exclusive]

$\bar{A}$ = complementary event of $A$ = "not $A$"

Ex1. Throw a die once. $\Omega = \{1,2,3,4,5,6\}$; $P(1) = \ldots P(6) = \frac{1}{6}$.

$A_1$ = "an even number turns up" $= \{2,4,6\}$

$A_2$ = "a prime number turns up" $= \{2,3,5\}$

$A_3$ = " ~~one~~ 1 turns up". $= \{1\}$

Then $P(A_1) \overset{\text{Prop. 3}}{=} \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.

$A_1 + A_2 = A_1 \cup A_2$ = "the number which turns up is either even <u>or</u> prime (or both)"
$= \{2,3,4,5,6\}$

$A_1 A_2 = A_1 \cap A_2$ = "the number which turns up is both even and prime" $= \{2\}$

$\bar{A}_1$ = "an odd number turns up" $= \{1,3,5\}$

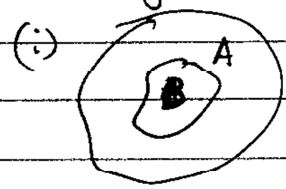
$\bar{A}_2$ = "a non-prime number turns up" $= \{1,4,6\}$

$\bar{A}_1 \cup \bar{A}_2 = \{1,3,4,5,6\} = \overline{A_1 \cap A_2}$ | $\bar{A}_1 \cap \bar{A}_2 = \{1\} = \overline{A_1 \cup A_2}$

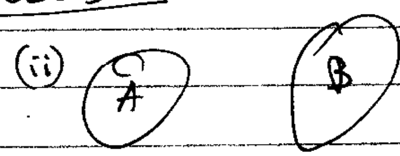
~~Disjoint Events~~

## 1.7.2 Conditional Probability.

In observing the outcomes of a random experiment, one is often interested in how the outcome of one event A is influenced by that of another event B. For example, in one extreme case, the relation between A and B may be such that A always occurs if B does,



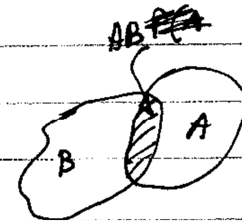

while in the other extreme case A never occurs if B does:




To characterize the relation between A and B, we introduce the

conditional probability of A given B
= prob. of A occurring ~~under the condition~~
if B is known to have occurred

$$\underbrace{P(A|B)}_{\text{Notation}} \underset{\substack{\uparrow \\ \text{definition}}}{=\!=} \frac{P(AB)}{P(B)},$$



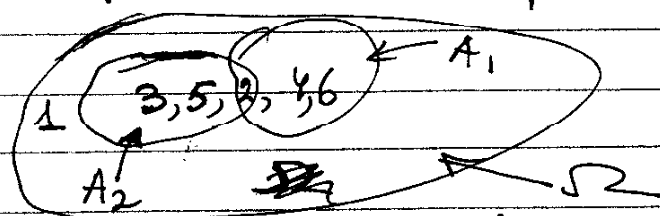

assuming $P(B) > 0$.

Ex 1 (continued).

Going back to our example,

$P(A_1|A_2) = ?$

P( ~~an~~ even number turned up, given that a prime number turned up) = ?




$\frac{1}{3}$, because it is one of three equiprobable outcomes

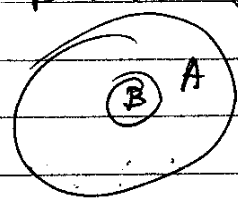
$$P(A_1|A_2) = \frac{P(A_1 A_2)}{P(A_2)} = \frac{P\{2\}}{P\{2,3,5\}} = \frac{1/6}{1/2} = \frac{1}{3}.$$

When we are conditioning on $A_2$, we are no longer operating in the whole $\Omega$. $A_2$ becomes the sample space; all probabilities are normalized by $P(A_2)$:

$$P(A_2|A_2) = \frac{P(A_2)}{P(A_2)} = 1.$$

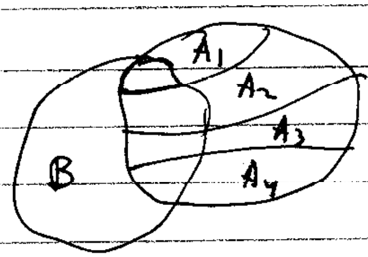
## Properties of Conditional Probabilities.

1. $0 \le P(A|B) \le 1$
2. If ~~A, B are~~ $AB = \varnothing$, then $P(A|B) = 0$
3. If $B \Rightarrow A$ (i.e., $B \subset A$), then $P(A|B) = 1$




4. If $A_1, A_2, \ldots$ are pairwise mutually exclusive with $A = \bigcup_{k=1}^{\infty} A_k$, then

$$P(A|B) = \sum_{k=1}^{\infty} P(A_k|B)$$




5. If $A \subset \bigcup_{k=1}^{\infty} B_k$, and $B_k$'s are mutually exclusive, then

$$A = \bigcup_{k=1}^{\infty} AB_k$$

$$\boxed{P(A) = \boxed{\sum_{k=1}^{\infty} P(AB_k)} = \sum_{k=1}^{\infty} P(A|B_k)P(B_k)}$$