1.7.2 Conditional Probability (continued).

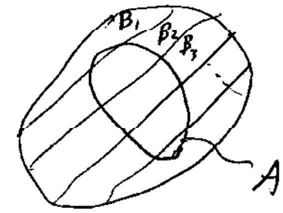
Lec. 20, Wed 10/10/01

Last time:

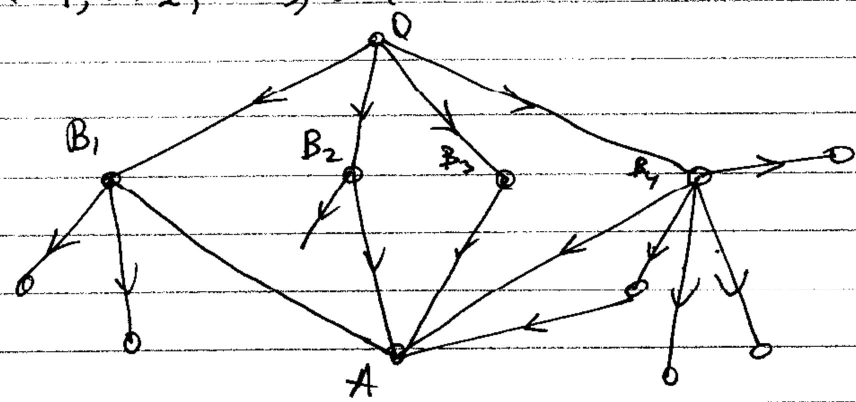
If $A \subset \bigcup_{k=1}^{\infty} B_k$, and B_k's are pairwise mutually exclusive, then

$$A = \bigcup_{k=1}^{\infty} AB_k, \text{ and so}$$

$$P(A) = \sum_{k=1}^{\infty} P(AB_k) = \sum_{k=1}^{\infty} P(A|B_k)P(B_k)$$



<u>Ex ~~1~~.</u> You leave point O, choosing one of the roads OB_1, OB_2, OB_3, OB_4 at random.



At every subsequent crossroads you again choose a road at random. What's the probability that you will arrive at A?

<u>Solution.</u>

$$P(\text{arrive at } B_k \text{ from } O) = \frac{1}{4} \quad \text{for } k = 1, 2, 3, 4.$$

$$P(\text{arrive at } A) = P(\text{go to } A \text{ from } B_1 \mid \text{arrived at } B_1 \text{ from } O) \cdot P(\text{arrive at } B_1 \text{ from } O) +$$

$$\cancel{+P(B_2 \to A \mid O \to B_2}$$

$$P(A \leftarrow B_2 \mid B_2 \leftarrow O)\, P(B_2 \leftarrow O)$$

$$+ P(A \leftarrow B_3 \mid B_3 \leftarrow O)\, P(B_3 \leftarrow O)$$

$$+ P(A \leftarrow B_4 \mid B_4 \leftarrow O)\, P(B_4 \leftarrow O) =$$

$$= \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + \frac{2}{5} \cdot \frac{1}{4}$$

$$= \frac{1}{4} \cdot \frac{10 + 15 + 30 + 12}{30} = \underline{\underline{\frac{67}{120}}}.$$

In other words, this formula often allows you to break seemingly intractable problems into simpler stages, and efficiently solve them. This basic principle is at the heart of Kalman filtering, and more general techniques ~~&~~ of estimation on graphs, as well as efficient coding algorithms.

~~[illegible]~~

1.7.3 Statistical Independence

Def. If $P(AB) = P(A)P(B)$, A and B are said to be statistically independent; otherwise, A and B are dependent.

"Independence" = occurrence of B has no influence on the probability of occurrence of A:

If A, B are independent, then

$$\boxed{P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)}$$

DNE

Ex. Let $A = \{$card picked at random from a full deck is a spade$\}$

$B = \{$ " " is a queen$\}$.

Are A, B indep.?

Solution. $P(A) = \frac{13}{52} = \frac{1}{4}$; $P(B) = \frac{4}{52} = \frac{1}{13}$

$P(AB) = P($Queen of spades$) = \frac{1}{52} = \frac{1}{4} \cdot \frac{1}{13} = P(A)P(B)$

Answer: yes, A and B are independent

Def. $A_1, A_2, \ldots, A_n$ are said to be (mutually) independent if

$$P(A_i A_j) = P(A_i)P(A_j)$$
$$P(A_i A_j A_k) = P(A_i)P(A_j)P(A_k)$$
$$\cdots$$
$$P(A_1 A_2 \ldots A_n) = P(A_1)P(A_2)\ldots P(A_n)$$

for <u>all</u> combinations of indices.

Ex. Throw a pair of dice.

$A_1 = \{$1st die turns up odd$\}$

$A_2 = \{$2nd die turns up odd$\}$

$A_3 = \{$total number is odd$\}$

1 2 3 4 5 6   1 2 3 4 5 6

Are these events pairwise ind.? Are they ind.?

Solution: $P(A_1) = P(A_2) = \frac{1}{2}$; A_1 & A_2 are independent

$P(A_3) = \frac{1}{2}$

Given that A_1 has occurred, A_3 occurs iff 2-nd die turns up even

$$P(A_3|A_1) = \frac{1}{2} = P(A_3) \Rightarrow A_1, A_3 \text{ are ind.}$$

Similarly,

$$P(A_3|A_2) = \frac{1}{2} = P(A_3) \Rightarrow A_2, A_3 \text{ are ind.}$$

<u>But</u> A_3 cannot occur if A_1 and A_2 both occur, and so

$$P(A_1 A_2 A_3) = 0,$$

whereas $P(A_1)P(A_2)P(A_3) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8} \neq P(A_1 A_2 A_3)$

$\Rightarrow A_1, A_2, A_3$ are <u>NOT</u> independent.

1.7.4 Random variables.

Def. A <u>random variable</u> X with values in the set E is a function which assigns a value $X(\omega) \in E$ to each outcome $\omega \in \Omega$. $X : \Omega \to E$.

not to be confused w/ frequency!

Most usual examples of $\boxed{E}$ are

non-negative integers	$\{0,1,2,\ldots\}$	X is a discrete random variable (i.e., can only assume values in a discrete set)
integers	$\{\ldots -1,0,1,\ldots\}$	
reals	$\mathbb{R}$	
non-negative reals	$\mathbb{R}^+$	

Ex. Flip a coin.

$$\Omega = \{H, T\}$$

(Two possible outcomes are "Heads" and "Tails")

Suppose

$$X(H) = 1$$
$$X(T) = -1$$

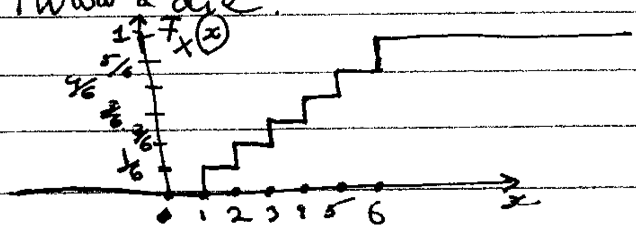
X is a random variable taking values in $E = \{1, -1\}$.

* a mapping from outcomes to numbers
* we will often analyze random variables without specifying the experiment which they came from. (However, you should always remember that a r.v. is a function from the sample sp. to numbers)
* will typically use capital letters $(X, Y, \ldots)$ to denote random variables.

Def. Cumulative distribution function (cdf) of a random variable X:

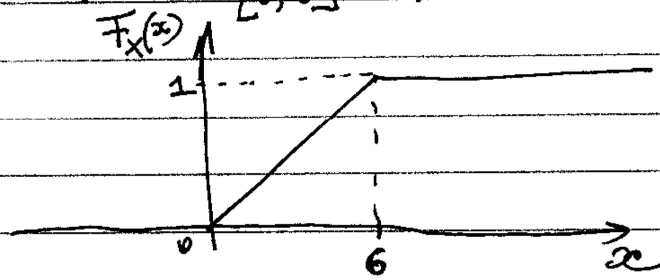
$$\underbrace{F_X(x)}_{\text{cdf of } X, \text{ evaluated at } x} = \underbrace{P\{X \le x\}}_{\text{is the probability that } X \le x}, \quad -\infty < x < \infty$$

Ex 1. Throw a die.



$X =$ ~~# of point~~ the number which turns up

Ex 2. toss a point ~~onto the~~ "at random" ~~at~~ onto the interval $[0, 6]$. $X =$ where it lands.



1. Note the distinction between the random variable X and the parameter of the cdf. It's not necessary to denote them by the same letter, e.g.,

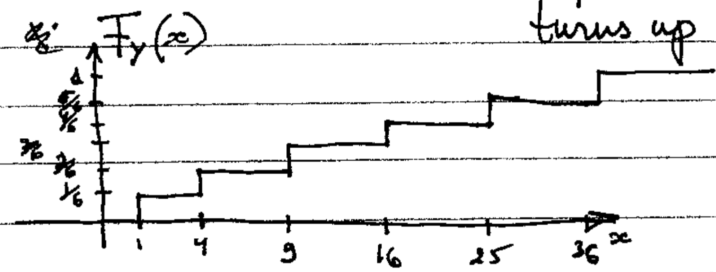
$F_X(\alpha)$ is legitimate notation meaning $\Pr(X \le \alpha)$.

2. Note also that notation

$f(X)$ is legitimate,

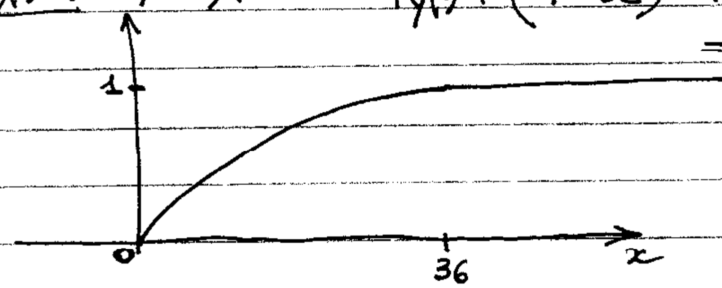
and means a random variable whose value at $\omega \in \Omega$ is $f(X(\omega))$.

Ex 1. $Y = X^2$ = square of the number which turns up



Ex 2. $Y = X^2$ $\quad F_Y(x) = P(Y \le x) = P(X^2 \le x) = P(X \le \sqrt{x})$

$= F_X(\sqrt{x})$



Properties of CDF:

- non-decreasing
- $\lim_{x \to \infty} F_X(x) = 1$
- $\lim_{x \to -\infty} F(x) = 0$

Probability density function (pdf):

$$f_X(x) = \frac{dF_X}{dx} \quad \text{(will also use } p_X(x)\text{)}$$

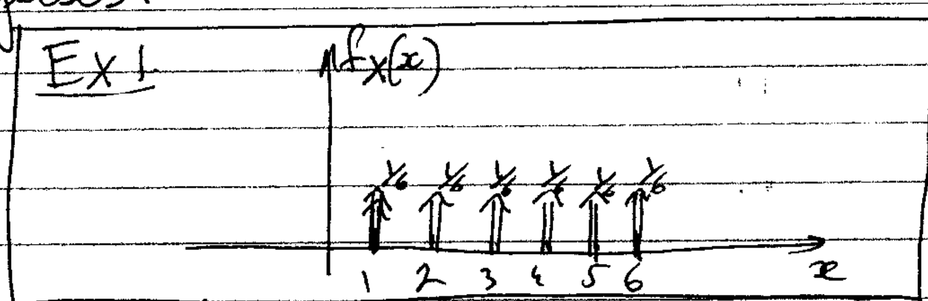
$$\int_a^b f_X(x)\,dx = F_X(b) - F_X(a) = P(X \le b) - P(X \le a)$$

$$\cancel{= P(X \le b) - [1 - P(X > a)] = P(\ \ X \le}$$

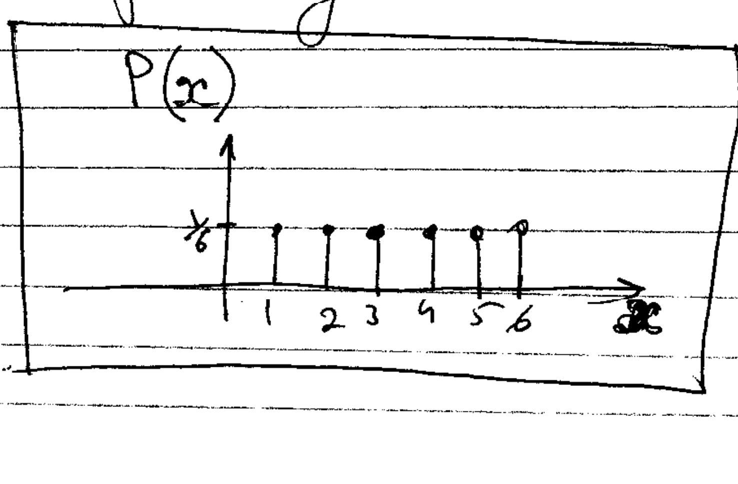
$$= 1 - P(X > b) - P(X \le a) = 1 - P(X > b \text{ or } X \le a)$$

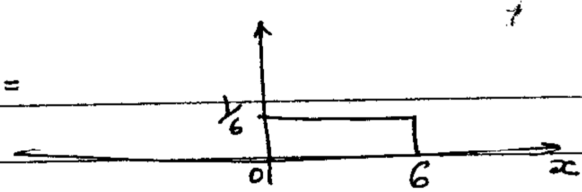
$$= P(a < X \le b)$$

If X is a discrete r.v., $f_X(x)$ contains impulses:



For discrete r.v.'s, it is sometimes more convenient to use the probability <u>mass</u> function:



Ex 2. $f_X(x) =$ 

Properties of pdf:

$$P(a < X \leq b) = \int_a^b f_X(x)\,dx$$

$$F_X(b) = \int_{-\infty}^{b} f_X(x)\,dx$$

$$f_X(x) \geq 0$$

$$\int_{-\infty}^{\infty} f_X(x) = 1$$