Lec 21 Fri 10/12/01.

Expectation (or Expected Value).

$$E\{g(X)\} \overset{\text{def}}{=} \int_{-\infty}^{\infty} g(x)\underbrace{f_X(x)}_{\text{pdf of } X}\,dx$$

e expected value of the rand. var. $g(X)$

(Note: when X is discrete, this is just $\sum_{x=-\infty}^{\infty} g(x)P(X=x)$)

Expected value of X: $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$

Second moment of X: $E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

Variance of X: $E\{(X-E(X))^2\}$

Expectation is <u>linear</u>, in the following sense:

$$E\{ag(X)+bh(X)+c\} = aE\{g(X)\} + bE\{h(X)\} + c$$

Therefore,

$$E\{(X-E(X))^2\} = E\{X^2 - 2XE(X) + (E(X))^2\}$$

$$= E\{X^2\} - 2(E(X))^2 + (E(X))^2 =$$

$$= E\{X^2\} - (E(X))^2$$

= 2-nd moment −

Ex. 1: Throw 1 die.

$$E(X) = \sum_{x=1}^{6} x\, P(X=x)$$

$$= 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + \ldots + 6\cdot\frac{1}{6}$$

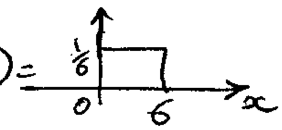
$$= \frac{21}{6} = \underline{3.5}$$

(Note $E(X)$ is not necessarily a value that X will assume with non-zero probability)

$$E(X^2) = (1\cdot 1 + 4\cdot 1 + 9\cdot 1 + 16\cdot 1 + 25 + 36)\cdot\frac{1}{6} = \frac{91}{6} = 15\frac{1}{6}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{3}{12}$$

Ex 2: X is uniform on $[0,6]$: $f_X(x) =$



$$E(X) = \int_0^6 \frac{1}{6} x\,dx = \frac{x^2}{12}\Big|_0^6 = 3$$

$$E(X^2) = \int_0^6 \frac{1}{6} x^2 dx = \frac{x^3}{18}\Big|_0^6 = 12$$

$$Var(X) = 12 - 3^2 = 3$$

1.7.5 Two Random Variables.

Def. Joint cumulative distribution function for two random variables, X and Y, is:

$$F_{X,Y}(x,y) = P(X \le x \text{ AND } Y \le y).$$

Joint pdf:

$$f_{XY}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(\alpha,\beta)\,d\beta\,d\alpha$$

The individual pdf's $f_X(x)$ and $f_Y(y)$ are then called <u>marginal</u> pdf's.

How can we get $f_X(x)$ and $f_Y(y)$ from $f_{X,Y}(x,y)$?

Note that

$$P(X \le x) = P(X \le x,\, Y \le \infty)$$

$$= F_{X,Y}(x,\infty)$$

$$= \int_{-\infty}^{x} \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(\alpha,\beta)\,d\beta}\,d\alpha \qquad (1)$$

On the other hand, we saw last time that

$$P(X \le x) = \int_{-\infty}^{x} \overbrace{f_X(\alpha)}\,d\alpha \qquad (2)$$

Identifying the integrands in (1) and (2), we see that

$$f_X(\alpha) = \int_{-\infty}^{\infty} f_{X,Y}(\alpha,\beta)\,d\beta,$$

or $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

Similarly,

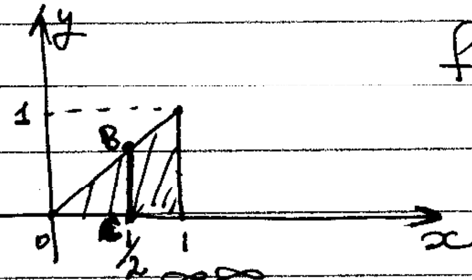
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Example 3. $f_{X,Y}(x,y) = \begin{cases} A, & 0 \le y \le x \le 1 \\ 0, & \text{else} \end{cases}$

(a) Find A

(b) Find $f_X\left(\frac{1}{2}\right)$.

Solution:



$f_{X,Y}(x,y) = A$ inside the triangle.

(a) To find A, $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$

$= A \cdot (\text{area of the triangle})$

$= A \cdot \frac{1}{2}$

$\Rightarrow \underline{A = 2}$.

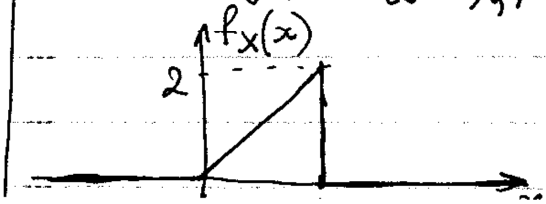
(b) $f_X\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} f_{X,Y}\left(\frac{1}{2}, y\right) dy$

$= \int_{0}^{1/2} 2 \cdot dy = 1$

I.e., in order to find the marginal density of X, integrate y out by computing the line integral of $f_{X,Y}$ along the line BC.

More generally,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^x 2 \cdot dy = 2x, & 0 \le x \le \\ 0, & \text{otherwise} \end{cases}$$



Def. Random variables X and Y are <u>independent</u> if
$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

(i.e., any event involving X is independent of any event involving Y.)

<u>Conditional density:</u>
$$f_{X|Y}(x|Y=y) \stackrel{\text{def}}{=} \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

If X, Y are independent, then
$$f_{X|Y}(x|y) = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x)$$
- i.e., the knowledge that $Y=y$ does not provide any information about X.

<u>Expected value:</u>
$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

(Note:
$$E(g(X)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x) f_{X,Y}(x,y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty} g(x) \left\{ \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right\} dx$$
$$= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx,$$
which is consistent with our previous definition)

<u>Correlation</u> of X and Y is $E(XY)$

<u>Covariance</u> of X and Y is
$$\lambda_{XY} = \mathrm{Cov}(X,Y) \stackrel{\text{def}}{=} E\{(X-E(X))(Y-E(Y))\} = E(XY) - E(X)E(Y).$$

Def. X and Y are <u>uncorrelated</u> if $\lambda_{XY} = 0$.

Remarks:

1. $\boxed{\text{Uncorrelated} \iff E(XY) = E(X)E(Y)}$

2. Independent $\Longrightarrow$ uncorrelated
(Because

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy \overset{\text{independence}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx \int_{-\infty}^{\infty} y f_Y(y)\,dy =$$

$$= E(X)E(Y))$$

3. Uncorrelated does not necessarily imply independent
(But if X, Y are, e.g. Gaussian, the two notions are equivalent.)

4. $Var(X) = Cov(X, X)$.

5. Correlation coefficient:

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

It is possible to show that $-1 \leq \rho_{XY} \leq 1$, with

$\rho_{XY} = 0 \Rightarrow X$ and Y are uncorrelated

$\rho_{XY} = 1 \Rightarrow X = aY + b$ with $a > 0$

$\rho_{XY} = -1 \Rightarrow X = aY + b$ with $a < 0$

Why, in addition to independence, do we need this other notion of uncorrelatedness? Because

~~it also tells us something~~

- also indicates to what extent X and Y are related (and in some cases is equivalent to independence)
- to determine whether X and Y are independent, we need to know $f_{XY}(x,y)$;
  to determine whether X and Y are uncorrelated, only need to know 1-st and 2-nd moments, which are easier to estimate (Lab 7, wk 1).