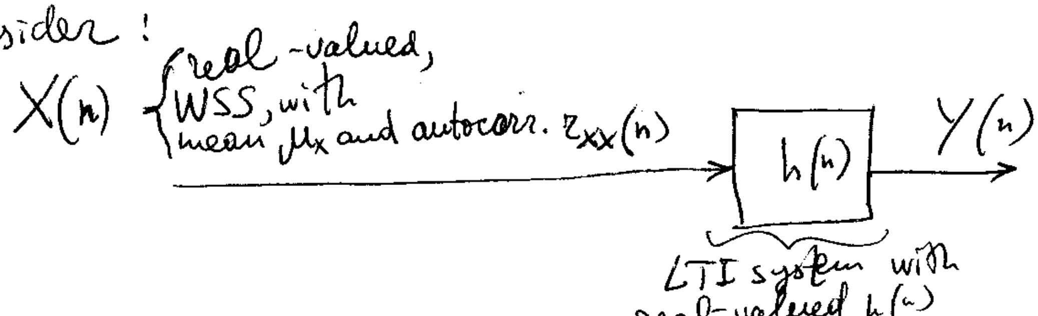
1.7.9 Filtering a Random Process (continued).

$r_{XX}(m,n) = E\{X(m)X(n)\}$ = autocorrelation

$C_{XY}(m,n)$ (or $r_{XY}(m,n)$) $= E\{X(m)Y(n)\}$ = cross-correlation.

What will happen to the mean and autocorrelation of a random process when it goes through an LTI system?

Consider:



(Recall: WSS means
(1) $E(X(n)) = \text{const}$
(2) $r_{XX}(m,n) = r_{XX}(0, n-m) \triangleq R_{XX}(n-m)$)

Given μ_X, $r_{XX}(n)$, $h(n)$, can we find $E(Y(n))$ and r_{YY}?

~~Complex? L.T.I. system, h(n) real-valued~~

~~$X(n) \to \boxed{h(n)} \to Y(n)$~~

~~How are statistics of X and Y related?~~

1. $$E(Y(n)) = E\left\{\sum_{m=-\infty}^{\infty} h(n-m) X(m)\right\}$$

$$\overset{\text{linearity of } E}{=} \sum_{m=-\infty}^{\infty} h(n-m) E(X(m))$$

$$\overset{\text{since } X(n) \text{ is WSS}}{=} \sum_{m=-\infty}^{\infty} h(n-m)\, \mu_x$$

$$= \mu_x \sum_{m=-\infty}^{\infty} h(m)$$

2. $$C_{XY}(m,n) = E\{X(m) Y(n)\}$$

$$= E\left\{X(m) \sum_{k=-\infty}^{\infty} h(n-k) X(k)\right\}$$

$$= \sum_{k=-\infty}^{\infty} h(n-k) E\{X(m) X(k)\}$$

$$= \sum_{k=-\infty}^{\infty} h(n-k)\, r_{XX}(k-m)$$

$\ell = k-m$
$\Rightarrow n-k = n-\ell-m$

$$\overset{}{=} \sum_{\ell=-\infty}^{\infty} h((n-m)-\ell)\, r_{XX}(\ell)$$

$$= (h * r_{XX})(n-m)$$ — ~~independent~~ only depends on $n-m$:

$$C_{XY}(m) = h * r_{XX}(n)$$

3. $r_{YY}(m,n) = E\{Y(m)Y(n)\} = E\{Y(m)\sum_{k=-\infty}^{\infty} h(n-k)X(k)\}$

$$= \sum_{k=-\infty}^{\infty} h(n-k)\, E\{X(k)Y(m)\}$$

$$= \sum_{k=-\infty}^{\infty} h(n-k)\, C_{XY}(m-k)$$

$$= \sum_{k=-\infty}^{\infty} h(n-k)\, C_{YX}(k-m)$$

$\ell = k-m$, $n-k = n-m-\ell$: $$= \sum_{\ell=-\infty}^{\infty} h((n-m)-\ell)\, C_{YX}(\ell)$$

$$= (h * C_{YX})(n-m)$$

$$= (h_- * C_{XY})(m-n),$$

where $h_-(n) = h(-n)$.

Again,

$r_{YY}(m,n) = r_{YY}(n,m) = R_{YY}(n-m)$

Also, found $E(Y(n)) = \text{const}$

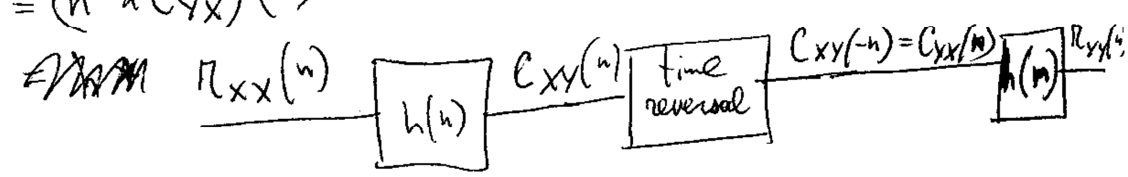
$\Rightarrow$ Y is WSS.

So, if the input to an LTI system is WSS, then the output is also WSS, with

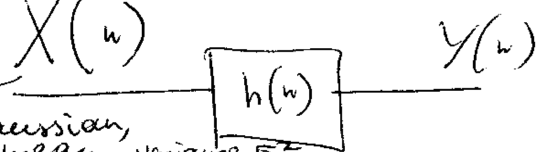
$$r_{YY}(n) = (h_- * C_{XY})(n)$$

$$= (h_- * h * r_{XX})(n)$$

$$= (h * C_{YX})(n)$$



Example



iid Gaussian, zero mean, variance σ_x^2.

$$Y(n) = X(n) + X(n-1)$$

Find $r_{YY}(n)$. – I.e., how correlated is the output with itself shifted by some lag value n?

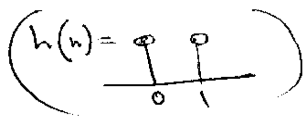
<u>Solution.</u> $r_{XX}(m) = E\{X(n)X(n+m)\}$

$$= \begin{cases} \sigma_x^2, & m=0 \\ 0, & m \neq 0 \text{ (since indep.)} \end{cases}$$

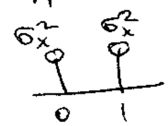
— white noise ($X(n)$ is uncorrelated with $X(m)$ unless $n=m$).

$$= \sigma_x^2 \delta(m)$$

$c_{XY}(m) = h * r_{XX}(m) =$



$=$

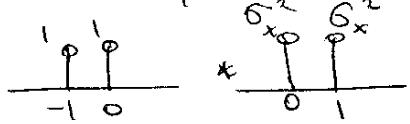


(Alternatively, $c_{XY}(m) = E\{X(n)Y(n+m)\}$

$$= E\{X(n)(X(n+m) + X(n+m-1))\}$$

$$= \begin{cases} \sigma_x^2 & \text{if } m=0 \text{ or } m=1 \\ 0 & \text{else (Because of indep.)} \end{cases}$$
)

$r_{YY}(m) = h_- * c_{XY}(m) =$



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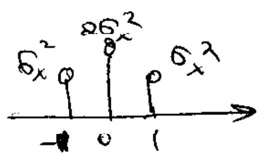
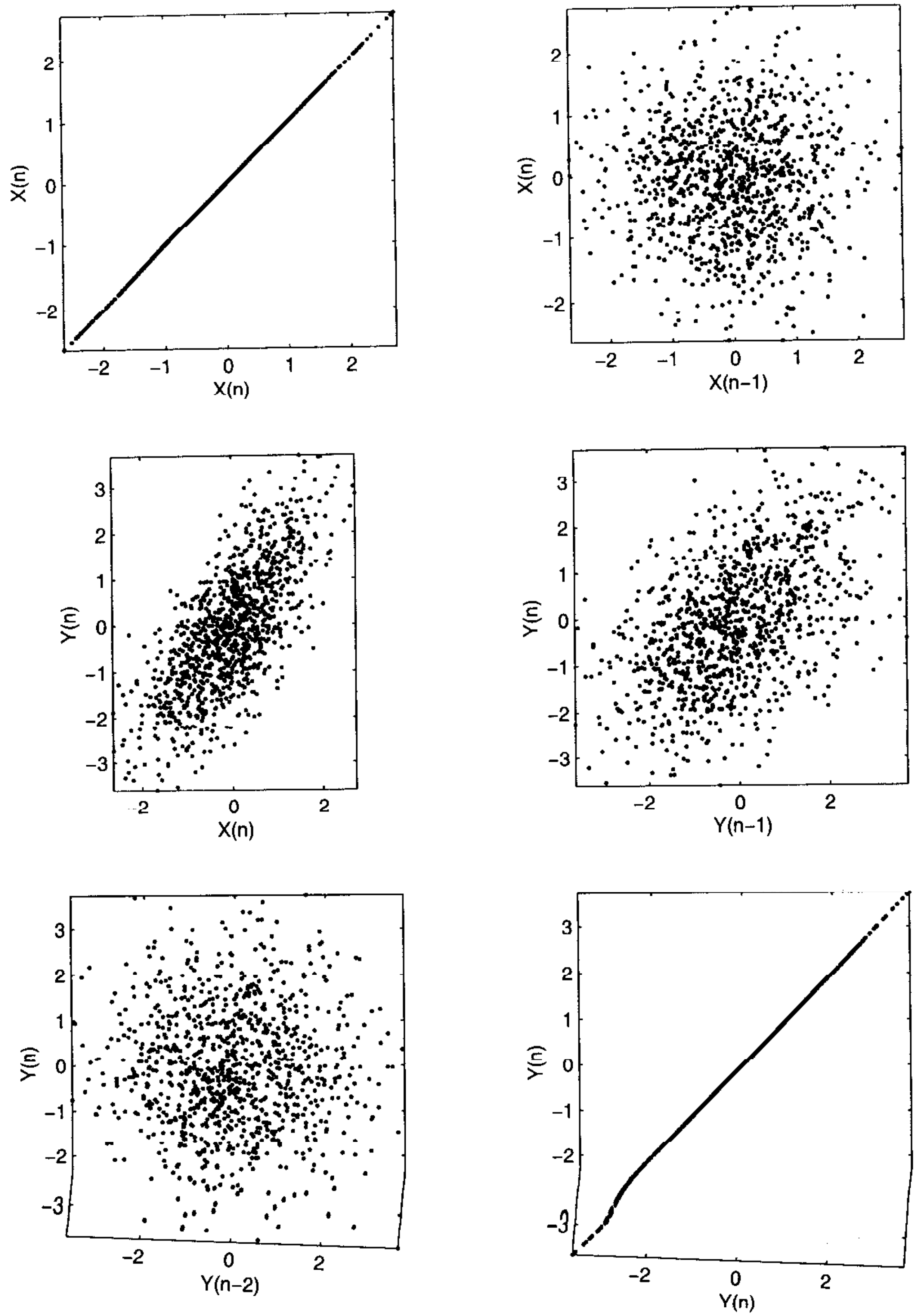


Is Y white?


X(n)
X(n)
X(n)
X(n−1)
Y(n)
X(n)
Y(n)
Y(n−1)
Y(n)
Y(n−2)
Y(n)
Y(n)

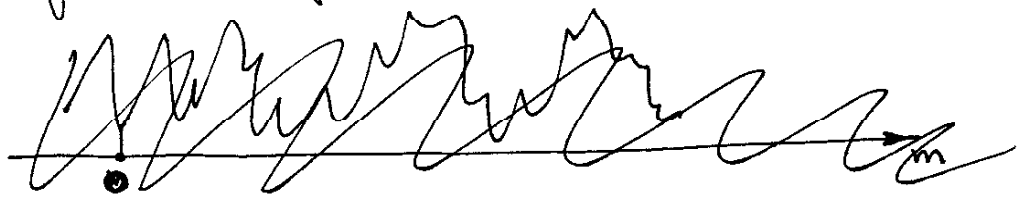
Estimating Correlation Functions (Lab 7).

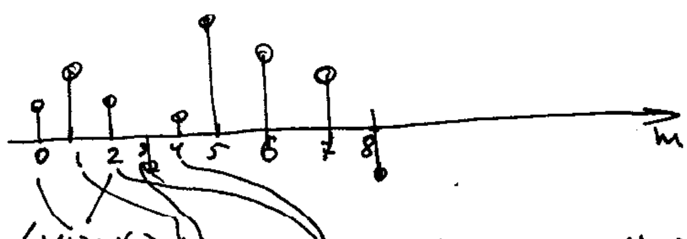
For wide-sense stationary processes,

autocorrelation $r_{XX}(n) = E[X(m)X(m+n)]$

cross-correlation $r_{XY}(n) = E[X(m)Y(m+n)]$

How to estimate $r_{XX}(m)$? Again, compute sample averages:





$\frac{1}{K}\left(X(0)X(2)+X(1)X(3)+X(2)X(4)+\ldots X(K-1)X(K+1)\right)$ = an estimate of $r_{XX}(2)$

More generally, if N points of $X(n)$ are available, form

$$\boxed{r'_{XX}(m) = \frac{1}{N-|m|}\sum_{n=0}^{N-|m|-1} X(n)X(n+|m|), \quad -(N-1) \le m \le N-1}$$

(Lab 7-2).

What is this useful for? Well, there is a lot of different applications, one of which is illustrated in Sect. 2 of the lab.

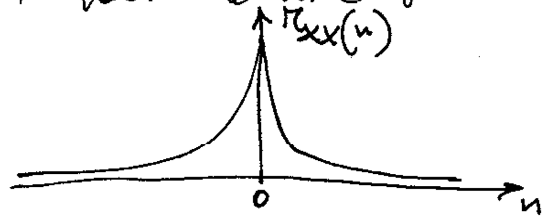
The key to this application is the following property of the autocorrelation function:

$$|r_{xx}(n)| \le r_{xx}(0)$$

And, in fact, for many random processes which model natural phenomena, the random variables become less and less correlated as they become more separated in time:

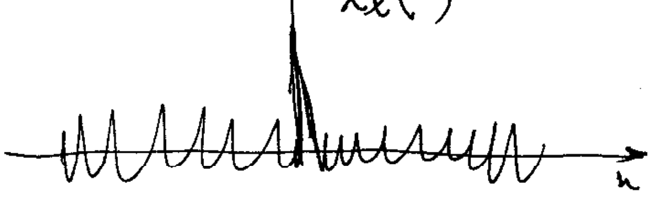
$$\lim_{n\to\infty} r_{xx}(n) = \left(E(X(n))\right)^2$$

So, a typical autocorrelation function might look like this:



(A random variable is very much correlated with itself, maybe somewhat correlated with its neighbors, and is basically uncorrelated from random variables which are far in the future or in the past)

Your estimate for Lab 7-2, Sect. [illegible], will look like this:



In radar or sonar, you transmit a signal $X(n)$, and then receive its reflection from an object, $Y(n)$. By measuring the delay between $X(n)$ and $Y(n)$, you can estimate the distance to the object.

$X(n)$ = transmitted

$Y(n)$ = received (attenuated, noisy, delayed version of $X(n)$).

Even though, what you do the lab, is an oversimplified example, you'll see that it's impossible to tell the delay just by looking at $X(n)$ and $Y(n)$. Any ideas as to how to estimate the delay by using correlation functions?

Estimate cross-correlation $C_{XY}(h)$, get