2.3 Recursive Estimation (continued).

Last time, we started solving the following problem:

$$S(n) = a\,S(n-1) + X(n)\ ,\ n \geq 1$$
$$Y(n) = S(n) + W(n)\ ,\ n \geq 0$$

As soon as $Y(n)$, find:

$\hat{S}_{n|n}$ = linear least-squares estimate of $S(n)$ based on $\{Y(0), Y(1), \dots, Y(n)\}$ = proj of $S(n)$ onto span$\{Y(0), Y(1), \dots, Y(n)\}$

$\lambda_{n|n} = \| S(n) - \hat{S}_{n|n} \|^2$

$\hat{S}_{n+1|n}$ = LLSE of $S(n+1)$ based on $\{Y(0), Y(1), \dots, Y(n)\}$

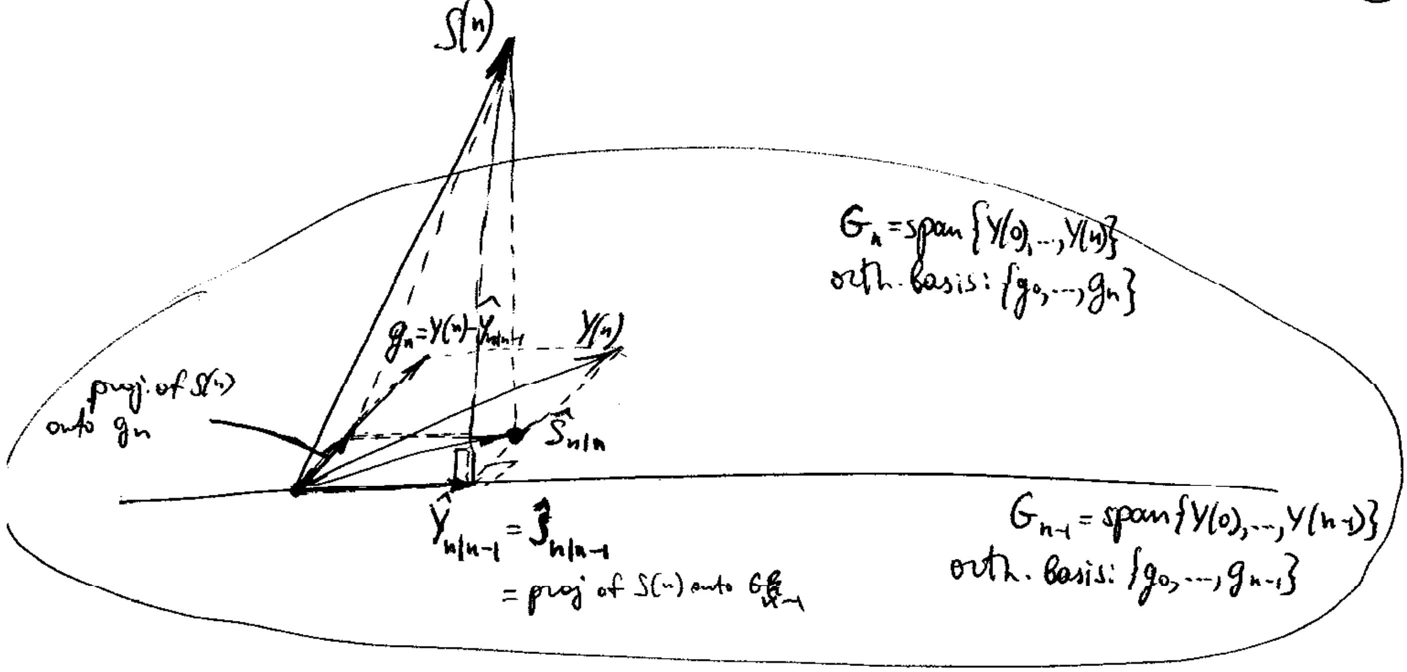
$\lambda_{n+1|n} = \| S(n+1) - \hat{S}_{n+1|n} \|^2$

Key idea: instead of trying to re-solve the normal equations every time, come up with a recursive algorithm.

Last time, we got:

$$\boxed{\begin{aligned} \hat{S}_{0|0} &= \frac{\lambda_0}{\lambda_0 + \lambda_W} Y(0) \\ \lambda_{0|0} &= \frac{\lambda_0 \lambda_W}{\lambda_0 + \lambda_W} \\ \hat{S}_{1|0} &= a \hat{S}_{0|0} \\ \lambda_{1|0} &= a^2 \lambda_{0|0} + \lambda_X \end{aligned}}$$

Now, assume we know $\hat{S}_{n|n-1}$ and $\lambda_{n|n-1}$, and try to get $\hat{S}_{n|n}$, $\lambda_{n|n}$, $\hat{S}_{n+1|n}$, and $\lambda_{n+1|n}$.



$$
\begin{aligned}
\hat{Y}_{n|n-1} &= \text{orth. proj of } Y(n) \text{ onto } G_{n-1} \\
&= \underbrace{\text{proj of } S(n) \text{ onto } G_{n-1}}_{\hat{S}_{n|n-1}} + \underbrace{\text{proj of } W(n) \text{ onto } G_{n-1}}_{\underline{0}} \\
&= \hat{S}_{n|n-1}
\end{aligned}
$$

Define $g_n = Y(n) - \hat{Y}_{n|n-1} = Y(n) - \hat{S}_{n|n-1}$

Then $g_n \perp G_{n-1}$, and $g_n \in G_n$

$\Rightarrow \{g_0, \dots, g_n\}$ is an orth. basis for G_n

$$
\begin{aligned}
\Rightarrow \hat{S}_{n|n} &= \underline{\text{proj. of } S(n) \text{ onto } g_n} + \underline{\text{proj of } S(n) \text{ onto } G_{n-1}} \\
&= \frac{\langle S(n), g_n \rangle}{\|g_n\|^2} g_n + \hat{S}_{n|n-1} \\
&= \frac{\langle S(n), Y(n) - \hat{S}_{n|n-1} \rangle}{\|Y(n) - \hat{S}_{n|n-1}\|^2} \left(Y(n) - \hat{S}_{n|n-1} \right) + \hat{S}_{n|n-1}
\end{aligned}
$$

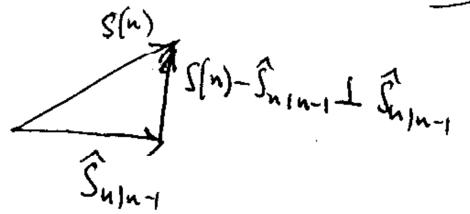
Note:

1. $$\langle S(n), \underbrace{Y(n)}_{S(n)+W(n)} - \hat{S}_{n|n-1} \rangle = \langle S(n), S(n) - \hat{S}_{n|n-1} + \cancel{W(n)} \rangle$$

$$= \langle S(n) - \hat{S}_{n|n-1} + \hat{S}_{n|n-1}, S(n) - \hat{S}_{n|n-1} \rangle$$

$$= \| S(n) - S_{n|n-1} \|^2 + \cancel{\langle \hat{S}_{n|n-1}, S(n) - \hat{S}_{n|n-1} \rangle} \to 0$$

$$= \lambda_{n|n-1}$$



2. $$\| \underbrace{Y(n)}_{S(n)+W(n)} - \hat{S}_{n|n-1} \|^2 = \| \underbrace{S(n) - \hat{S}_{n|n-1}, + W(n)}_{\text{indep.}} \|^2 = \| S(n) - \hat{S}_{n|n-1} \|^2 + \| W(n) \|^2 =$$

$$= \lambda_{n|n-1} + \lambda_w \qquad (*)$$

So,

$$\boxed{\hat{S}_{n|n} = \hat{S}_{n|n-1} + \frac{\lambda_{n|n-1}}{\lambda_{n|n-1} + \lambda_w} (Y(n) - \hat{S}_{n|n-1})} \qquad (u1)$$

From the right triangle $\hat{S}_{n|n-1} - S(n) - \hat{S}_{n|n}$, we have:

$$\lambda_{n|n} = \lambda_{n|n-1} - \| \hat{S}_{n|n} - \hat{S}_{n|n-1} \|^2$$

$$= \lambda_{n|n-1} - \left\| \frac{\lambda_{n|n-1}}{\lambda_{n|n-1} + \lambda_w} (Y(n) - \hat{S}_{n|n-1}) \right\|^2$$

$$\overset{\text{from } (*) \text{ above}}{=} \lambda_{n|n-1} - \frac{\lambda_{n|n-1}^2}{(\lambda_{n|n-1} + \lambda_w)^2} (\lambda_{n|n-1} + \lambda_w)$$

$$\boxed{= \lambda_{n|n-1} - \frac{\lambda_{n|n-1}^2}{\lambda_{n|n-1} + \lambda_w}} \qquad (u2)$$

measurement update step of the Kalman filter

$$\hat{S}_{n+1|n} = \text{proj of } \overbrace{S(n+1)}^{aS(n)+X(n+1)} \text{ onto } \mathcal{G}_n$$
$$= \text{proj of } aS(n) \text{ onto } \mathcal{G}_n + \underbrace{\text{proj of } X(n+1) \text{ onto } \mathcal{G}_n}_{0}$$
$$\boxed{= a\,\hat{S}_{n|n}} \qquad (p1)$$

$$\lambda_{n+1|n} = \| \underbrace{S(n+1)}_{aS(n)+X(n+1)} - \underbrace{\hat{S}_{n+1|n}}_{a\hat{S}_{n|n}} \|^2 = \| aS(n) - a\hat{S}_{n|n} + X(n+1) \|^2$$
$$= \| a(S(n) - \hat{S}_{n|n}) \|^2 + \| X(n+1) \|^2$$
$$\boxed{= a^2 \lambda_{n|n} + \lambda_x} \qquad (p2)$$

prediction step of the Kalman filter

Remarks.

1. Note: we can start the recursion with

$$\begin{cases} \hat{S}_{0|-1} = 0 & \text{– i.e., in the absence of any data, our best guess for } S(0) \text{ is } E(S(0)). \\ \lambda_{0|-1} = \lambda_0 \end{cases}$$

Then
$$\left.\begin{aligned} \hat{S}_{0|0} &= 0 + \frac{\lambda_0}{\lambda_0+\lambda_w}\left(Y(0) - 0\right) \\ \lambda_{0|0} &= \lambda_0 - \frac{\lambda_0^2}{\lambda_0+\lambda_w} = \frac{\lambda_0\lambda_w}{\lambda_0+\lambda_w} \end{aligned}\right\} \text{ same as what we got before.}$$

2. $g_0, g_1, g_2, \ldots$ is called the innovations process. We obtained it ~~using the~~ from $Y(0), Y(1), \ldots$ using the Gram-Schmidt orthogonalization procedure (HW 13 Prob 2)

3. $$\lambda_{n|n} = \lambda_{n|n-1} - \underbrace{\frac{\lambda_{n|n-1}^2}{\lambda_{n|n-1}+\lambda_w}}_{\text{the amount of error variance that got removed by incorporating the additional observation } Y(n)} = \frac{\lambda_{n|n-1}\lambda_w}{\lambda_{n|n-1}+\lambda_w} = \frac{1}{\lambda_{n|n-1}^{-1}+\lambda_w^{-1}}$$

$$\lambda_{n|n}^{-1} = \lambda_{n|n-1}^{-1} + \lambda_w^{-1}$$

4. Eq. (u2) and (p2) are called the DT Riccati equations.

4. Instead of solving multiple systems of linear equations of growing size, we got a very low complexity recursive algorithm, which produces not only the estimates, but also the corresponding error variances.
Note also that the error variances $\lambda_{n|n}$, $\lambda_{n|n-1}$ do not depend on the data and can therefore be pre-computed.

~~5.~~ $\cancel{\hat{S}_{n|n} = a\hat{S}_{n-1|n-1} + \frac{a^2\lambda_{n-1|n-1}+\lambda_x}{a^2\lambda_{n-1|n-1}+\lambda_x+\lambda_w}\left(Y(n) - a\hat{S}_{n-1|n-1}\right)}$

5.
$$\left.\begin{aligned} \lambda_{n|n} &= \frac{1}{(a^2\lambda_{n-1|n-1}+\lambda_x)^{-1}+\lambda_w^{-1}} \\ \hat{S}_{n|n} &= a\hat{S}_{n-1|n-1} + \frac{\lambda_{n|n}}{\lambda_w}\left(Y(n) - a\hat{S}_{n-1|n-1}\right) \end{aligned}\right)$$
updated form of the Kalman filter

$$\left.\begin{aligned} \lambda_{n+1|n} &= \frac{a^2}{\lambda_{n|n-1}^{-1}+\lambda_w^{-1}} + \lambda_x \\ \hat{S}_{n+1|n} &= a\hat{S}_{n|n-1} + \frac{a\lambda_{n|n-1}}{\lambda_{n|n-1}+\lambda_w}\left(Y(n) - \hat{S}_{n|n-1}\right) \end{aligned}\right)$$
predictor form of the Kalman filter