Lec 34: Review for Exam 3   11/14/01

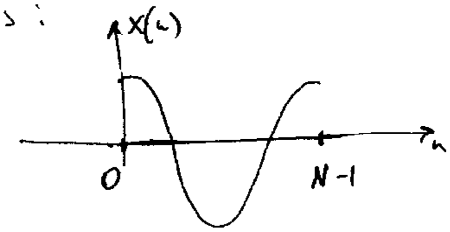
Why circular convolution?

- Because DFT computes it – i.e.,

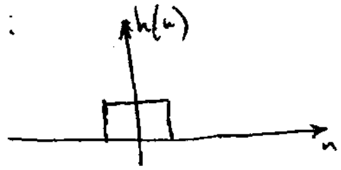
  $$DFT^{-1}\{DFT(x)\,DFT(y)\} = x \circledast y$$

- It sometimes leads to less significant edge effects

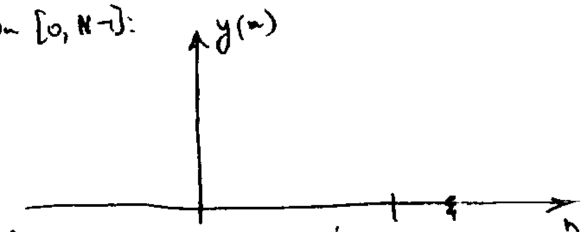
  e.g., filter this:



with this:



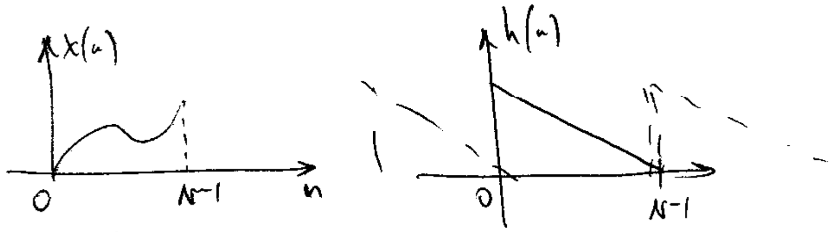
want $y(n)$ on $[0, N-1]$:

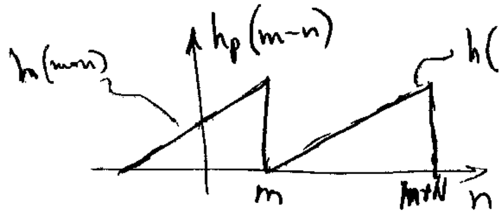


circular convolution ⟺ periodic extension
(even better: symmetrically reflect, then extend periodically)

If x, h are duration N, then $y_p(n) = x \; N \; h(n)$ is obtained as follows: ②

- periodically extend one of the signals,
- flip & slide





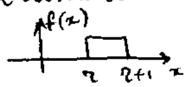
$$y_p(m) = \underbrace{\sum_{n=-\infty}^{\infty} h_p(m-n)\,x(n)}_{y(m),} + \underbrace{\sum_{n=-\infty}^{\infty} h(m+N-n)\,x(n)}_{y(m+N)}$$

if $m=N-1$, this is 0.

where $y = x * h$

$$= \begin{cases} y(m) + y(m+N), & m = 0, 1, \dots, N-2 \\ y(N-1), & m = N-1 \end{cases}$$

Important ideas in probability and random sequences:

- Given observations of iid random variables, we can estimate parameters of their distribution (such as m, σ, r in , etc) or the whole distribution

- If Y and X are <u>dependent</u>, then observing Y will tell us something about X

- If $X(n) \rightarrow \boxed{LTI} \rightarrow Y(n)$ $\Rightarrow$ can tell a lot about Y from the knowledge of X and the system

Example 1. a) Suppose X, Y are independent, with pdf's f_X, f_Y. What's the pdf of $Z = X+Y$? [Answer: $f_Z(z) = \int_{-\infty} f_X(z-y) f_Y(y)\,dy$]

b) If X, Y are indep., discrete, with pmf p_X and p_Y, what's the pmf of $Z = X+Y$? [Answer: $p_Z(z) = \sum_{y=-\infty} p_X(z-y) p_Y(y)$]

Solution. (a)

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \text{Prob}(Z \le z) = \frac{d}{dz} \text{Prob}(X+Y \le z)$$

$$= \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x,y)\,dx\,dy = \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y)\,dx\,dy$$

$$= \frac{d}{dz} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{z-y} f_X(x)\,dx \right\} f_Y(y)\,dy = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy$$

(b)

$$p_Z(z) = \text{Prob}(X+Y=z) = \sum_{y=-\infty}^{\infty} \underbrace{\text{Prob}(X = z-y \mid Y=y)\,\text{Prob}(Y=y)}_{\text{Prob}(X = z-y \text{ AND } Y=y)}$$

$$\overset{\text{indep.}}{=} \sum_{y=-\infty}^{\infty} \text{Prob}(X = z-y)\,\text{Prob}(Y=y)$$

$$= \sum_{y=-\infty}^{\infty} p_X(z-y)\, p_Y(y)$$

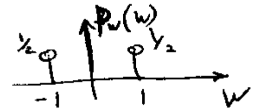
Example 2. Suppose X, Y are independent, with cdf's F_X, F_Y. What's cdf of $Z = \max(X,Y)$? [Answer: $F_X(z) F_Y(z)$]

Solution.

$$F_Z(z) = \text{Prob}(\max(X,Y) \le z) = \text{Prob}(X \le z \text{ and } Y \le z)$$

$$= \text{Prob}(X \le z)\,\text{Prob}(Y \le z) = F_X(z) F_Y(z).$$

HW 11 Prob 3. Let $W(n)$ be iid, with pmf



Let $X(n) = \sum_{m=1}^{n} W(m)$ for $n \geq 1$, and $X(0) = 0$.

Denote $p_n(k) = \text{Prob}(X(n) = k)$ = pmf of $X(n)$.

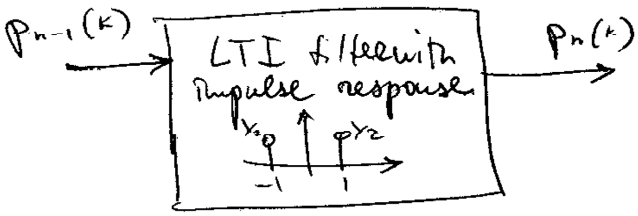
(a) Relate p_n and p_{n-1}.

Since $X(n) = X(n-1) + W(n)$, and since $X(n-1)$ & $W(n)$ are indep.,

$$\text{pmf}(X(n)) = \text{pmf}(X(n-1)) * \text{pmf}(W(n))$$

$$p_n(k) = p_{n-1} * f_w(k)$$

$$= \tfrac{1}{2} p_{n-1}(k-1) + \tfrac{1}{2} p_{n-1}(k+1)$$



(b) Find $p_n(k)$.

$$p_n(k) = \underbrace{p_w * p_w * \cdots * p_w}_{n \text{ terms}}(k)$$

$$p_1 = p_w \rightarrow \boxed{p_w} \xrightarrow{p_2} \boxed{p_w} \xrightarrow{p_3} \cdots \rightarrow \boxed{p_w} \xrightarrow{p_n}$$

Z-tr:

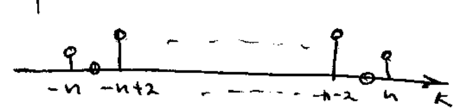
$$P_n(z) = \left(P_w(z)\right)^n = \left\{\tfrac{1}{2}\left(z + z^{-1}\right)\right\}^n$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{m=0}^{n} \binom{n}{m} z^{n-m} z^{-m}$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{m=0}^{n} \binom{n}{m} z^{-(2m-n)}$$

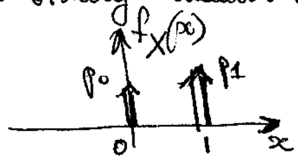
$$p_n(k) = \left(\tfrac{1}{2}\right)^n \sum_{m=0}^{n} \binom{n}{m} \delta\left(k - [2m-n]\right)$$

$$p_n(k) = \begin{cases} \left(\tfrac{1}{2}\right)^n \binom{n}{\frac{n+k}{2}}, & 0 \leq \frac{n+k}{2} \leq n \text{ and } n+k \text{ is even} \\ 0 & \text{else} \end{cases}$$



Suppose we are given:

X is a binary random variable, with



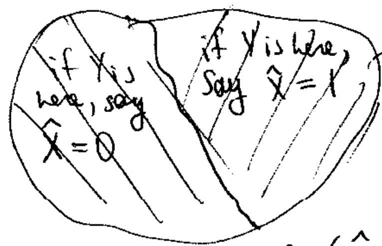
} prior model

Y is a random variable, and $f_{Y|X}(y|x)$ is known. } observation model

Estimate X from Y, to minimize the probability of error.

$$\mathrm{Prob}(\hat{X}(Y) \neq X) = \mathrm{Prob}(\hat{X}(Y)=1 \mid X=0)\,\mathrm{Prob}(X=0)$$
$$+ \mathrm{Prob}(\hat{X}(Y)=0 \mid X=1)\,\mathrm{Prob}(X=1)$$

$$= p_0 \int_{\text{say } 1} f_{Y|X}(y|0)\,dy + p_1 \int_{\text{say } 0} f_{Y|X}(y|1)\,dy$$



At every point y, one integral is on, one is off.

Pick $\hat{X}=1$ for those values of y for which the first integrand is smaller than the second one; pick $\hat{X}=0$ for those y's for which the second integrand is smaller.

To minimize $\mathrm{Prob}(\hat{X}(Y) \neq Y)$,

if $p_0 f_{Y|X}(y|0) < p_1 f_{Y|X}(y|1)$ say 1

if $p_0 f_{Y|X}(y|0) > p_1 f_{Y|X}(y|1)$ say 0

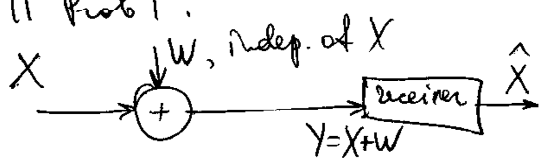
Note: when $p_0 = \frac{1}{2}$, this is the same as ML.

Since $\mathrm{Prob}(X=1|Y=y) = \dfrac{f_{Y|X}(y|1)\,p_1}{f_Y(y)}$

$\mathrm{Prob}(X=0|Y=y) = \dfrac{f_{Y|X}(y|0)\,p_0}{f_Y(y)}$,

this rule is equivalent to $\mathrm{Prob}(X=1|Y=y) \underset{\text{say } 0}{\overset{\text{say } 1}{\gtrless}} \mathrm{Prob}(X=1|Y=y)$.

HW 11 Prob 1:



$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}}$$

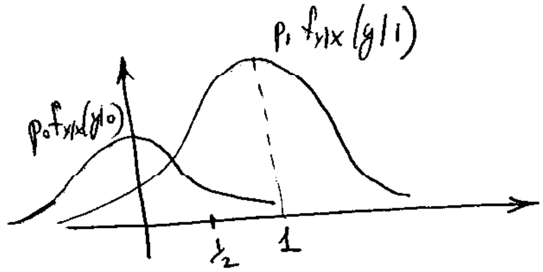
MPE Decision rule:

$$\frac{p_0}{\cancel{\sqrt{2\pi}\,\sigma}} e^{-\frac{y^2}{2\sigma^2}} \underset{1}{\overset{0}{\gtrless}} \frac{p_1}{\cancel{\sqrt{2\pi}\,\sigma}} e^{-\frac{(y-1)^2}{2\sigma^2}}$$

$$\ln p_0 - \frac{y^2}{2\sigma^2} \underset{1}{\overset{0}{\gtrless}} \ln p_1 - \frac{(y-1)^2}{2\sigma^2}$$

$$2\sigma^2 \ln\frac{p_0}{p_1} \underset{1}{\overset{0}{\gtrless}} 2y-1$$

$$y \underset{0}{\overset{1}{\gtrless}} \underbrace{\sigma^2 \ln\frac{p_0}{p_1} + \frac{1}{2}}_{\text{threshold } \eta}$$



Remarks:

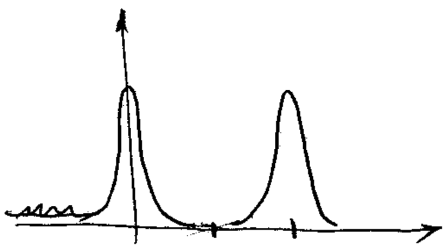
1. $p_0 < p_1 \Rightarrow \ln\frac{p_0}{p_1} < 0$

$\Rightarrow \eta < \frac{1}{2}$

$p_0 > p_1 \Rightarrow \eta > \frac{1}{2}$

$p_0 = p_1 \Rightarrow \eta = \frac{1}{2}$

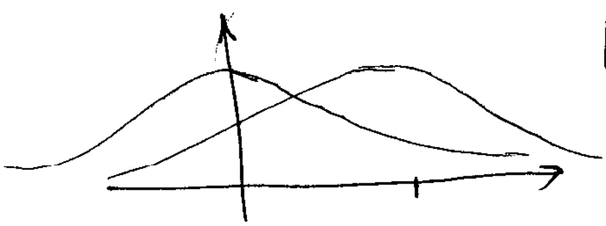
2.



small $\sigma \Rightarrow$ small overlap

$\Rightarrow$ ~~small~~ few errors

$\sigma \to 0 \Rightarrow P(\text{error}) \to 0$



large $\sigma \Rightarrow$ large overlap

$\Rightarrow$ lots of errors

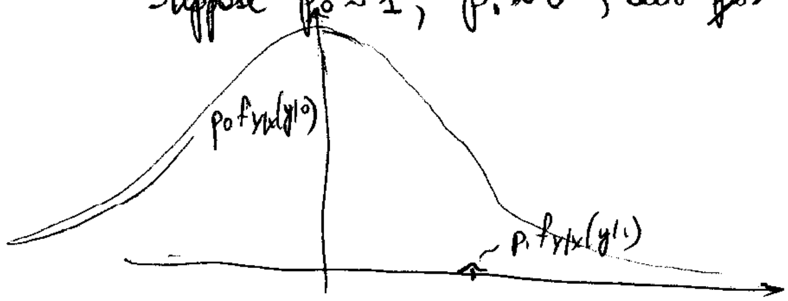
$\sigma \to \infty \Rightarrow P(\text{error}) \to \frac{1}{2}$

3. "Optimal" does not mean "good"!

Suppose $p_0 \approx 1$, $p_1 \approx 0$, but "$X=0$" = "House is on fire"

$\Rightarrow$ practically always decide 1.

$\left(\eta = \sigma^2 \ln\left(\frac{p_0}{p_1}\right) + \frac{1}{2} \text{ is } \underline{\underline{\text{huge}}}\right)$



Overall probability of ~~exact decision~~ error

$$p_0 \int_{\tau}^{\infty} f_{Y|X}(y|0)\,dy + \underbrace{p_1}_{\approx 0} \int_{-\infty}^{\tau} f_{Y|X}(y|1)\,dy \approx 0$$

$\tau \approx \infty$

Conditional probability of error given $X=1$:

$$\text{Prob}\left(\hat{X}(Y)=0 \mid X=1\right) = \int_{-\infty}^{\tau \approx \infty} f_{Y|X}(y|1)\,dy \approx \int_{-\infty}^{\infty} f_{Y|X}(y|1)\,dy = 1$$