3.3 Multiscale Analysis.

Example 1 (random walk in 1-D).

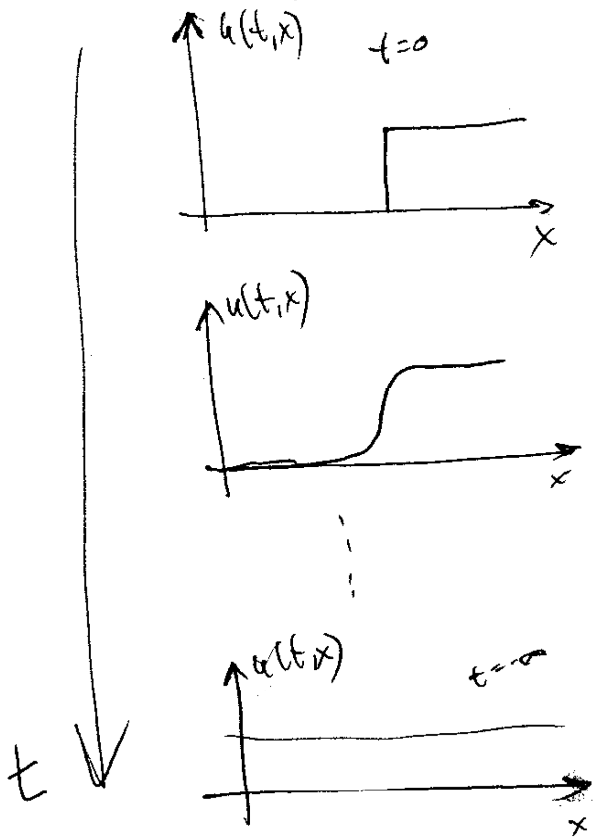
$$g^{(i+1)}(n) = \underbrace{g^{(i)}(n)}_{\text{prob. mass fcn of position after i-th step.}} + \alpha\left\{\left[g^{(i)}(n-1)-g^{(i)}(n)\right]+\left[g^{(i)}(n+1)-g^{(i)}(n)\right]\right\}$$

Example 1a (Brownian motion).

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \text{where} \quad u(t,x) = \text{pdf of position at time } t$$

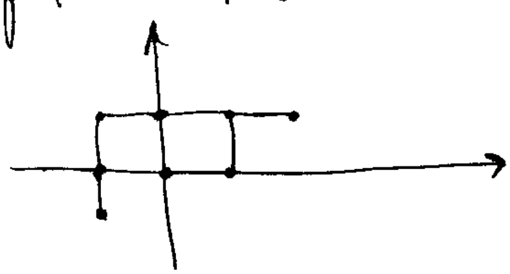
The pdf of each particle evolves according ~~the~~ to this PDE.
This equation also governs the diffusion processes

E.g., time $t=0$



Remarks:

1. Same partial differential equation governs the diffusion of heat — i.e., if  is the initial temperature distribution in the room.
2. If the initial distribution is $u(0,x) = u^{(0)}(x)$, then
$$u(t,x) = u^{(0)} * G^{(t)}(x),$$
where $G^{(t)}(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$ (a Gaussian with variance $2t$).
(Exercise in differentiation)
3. The iteration number (i in the discrete setting, t in the continuous setting) is called scale:
"larger t" = "coarser scale"
"smaller t" = "finer scale",
to reflect the fact that $u(t,x)$ is a "coarsened version" of $u(0,x)$.
4. The aggregation of the resulting signals ($\{g^{(i)}(n)\}_{i=0}^{\infty}$ or $\{u(t,x)\}_{t=0}^{\infty}$) is called a scale-space. This linear scale space, as we have seen, is not very good for noise removal. [also, multiscale analysis, multiresolution analysis]
Perona-Malik scale-space:
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} F\left(\frac{\partial u}{\partial x}\right),$$
or
$$g^{(i+1)}(n) = g^{(i)}(n) + \alpha\{F[g^{(i)}(n+1) - g^{(i)}(n)] + F[g^{(i)}(n-1) - g^{(i)}(n)]\},$$
where $F(v) =$ 
5. Same analogy works for images in 2-D.
Random walk in a plane: move E, W, N, S, with prob. α each; stay put with prob. $1-4\alpha$ $(0 \le \alpha \le \frac{1}{4})$.



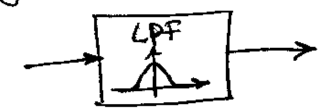
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u$$

$$\frac{\partial u}{\partial t} = \vec{\nabla} \cdot F(\nabla u)$$

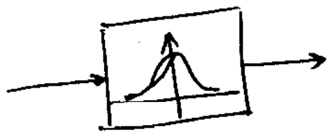
or $g^{(i+1)}(m,n) = \ldots$ — see prev. class

Let us now go back to the linear case, and consider the following question: can we reconstruct the signal from its coarsened versions?

In general, no:

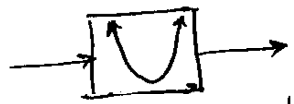


high frequencies are lost $\Rightarrow$ inverse does not exist



- no stable reconstruction

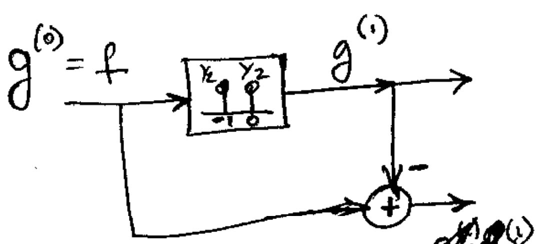
(Because the inverse is



$\Rightarrow$ a very small error in high freq. will lead to poor reconstr.)

In addition, inverse of an FIR filter is an IIR filter.

What if we retain some additional information?



We can certainly reconstruct $g^{(0)}$ from $g^{(1)}$ and $h^{(1)}$!

$$g^{(0)} = h^{(1)} + g^{(1)}$$

Do we need ~~the~~ all the samples in $g^{(1)}$ and $h^{(1)}$ to reconstruct $g^{(0)}$?

$$\left.\begin{aligned} g^{(1)}(0) &= \tfrac{1}{2} g^{(0)}(0) + \tfrac{1}{2} g^{(0)}(1) \\ h^{(1)}(0) &= g^{(0)}(0) - g^{(1)}(0) = \tfrac{1}{2} g^{(0)}(0) - \tfrac{1}{2} g^{(0)}(1) \end{aligned}\right\} \Rightarrow$$

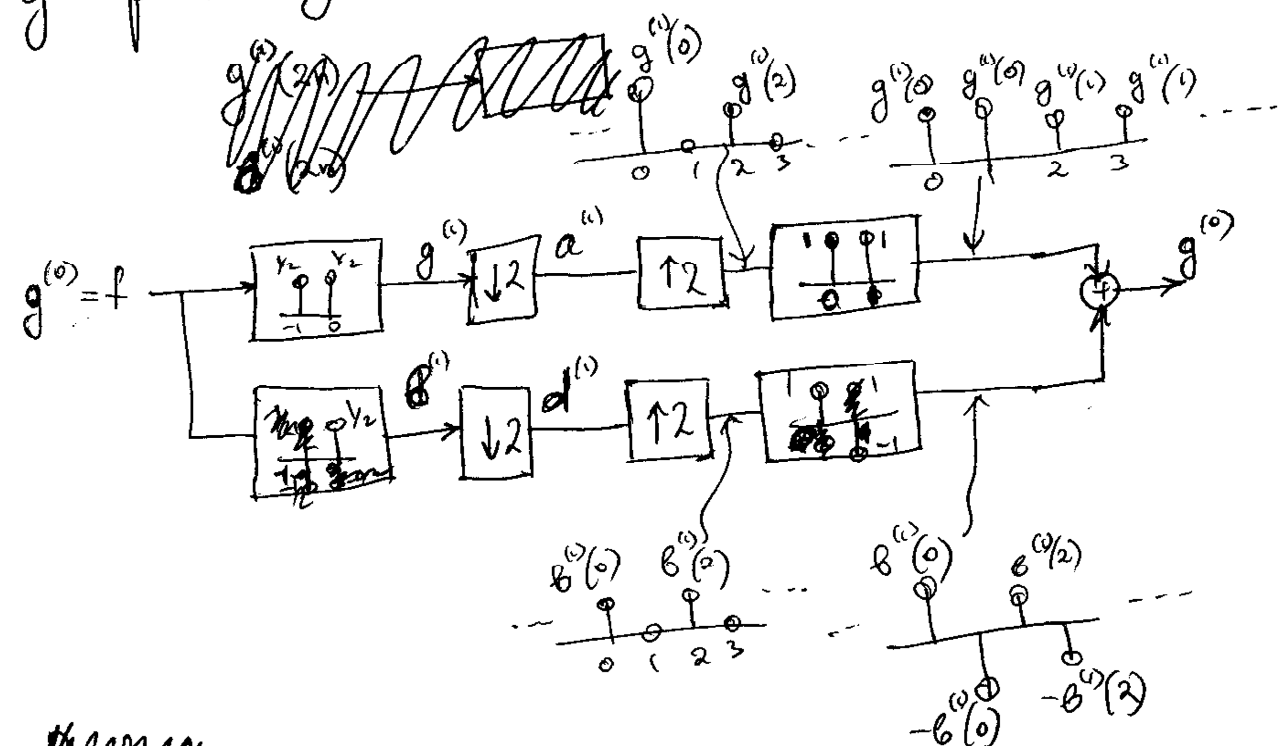
$$\Rightarrow \begin{cases} g^{(0)}(0) = g^{(1)}(0) + h^{(1)}(0) \\ g^{(0)}(1) = g^{(1)}(0) - h^{(1)}(0) \end{cases}$$

$\Rightarrow$ we can reconstruct the 0-th and 1-st samples of $g^{(0)}$ just from the 0-th samples of $g^{(1)}$ and $h^{(1)}$.

Similarly,

$$\begin{cases} g^{(1)}(2) = \frac{1}{2}g^{(0)}(2) + \frac{1}{2}g^{(0)}(3) \\ d^{(1)}(2) = \frac{1}{2}g^{(0)}(2) - \frac{1}{2}g^{(0)}(3) \end{cases} \Rightarrow \begin{cases} g^{(0)}(2) = g^{(1)}(2) + d^{(1)}(2) \\ g^{(0)}(3) = g^{(1)}(2) - d^{(1)}(2) \end{cases}$$

...and so on. We can therefore throw away every other sample of $g^{(1)}$ and $d^{(1)}$, and still reconstruct $g^{(0)}$ perfectly:



Therefore,