1 Lecture 4.

Last time, we started talking about systems and properties of systems.

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                 ___________
x               |           |     y=S[x]
------>| system S |------>
                |___________|
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A system transforms a signal into another signal. Very often, systems are specified by input-output relationships, for example,

input-output view

$$y(n) = \sum_{k=0}^{n} x(k) \text{ for all integer } n; \quad (1)$$

$$y(n) = 2x(n) + 3 \text{ for all integer } n. \quad (2)$$

This specification of the domain is important, because the input to the system is the whole signal x, for all points in its domain of definition, and the output is the whole signal y, for all points in its domain of definition.

behavioral view

$$q(n) = 3\,q(n-1) \quad (3)$$

A system is anything that imposes constraints.
A behavior is a signal [or a set of signals] that satisfies the constraints.

1.2 Systems.

1.2.2 Properties of Systems.

a) Linearity. A system S is defined to be *linear* if, for any two input signals x_1 and x_2, and for any two numbers a_1 and a_2, it satisfies:

$$y_3(n) = a_1 y_1(n) + a_2 y_2(n) \text{ for all } n,$$

where y_1, y_2, and y_3 are the responses of the system to x_1, x_2, and $a_1 x_1 + a_2 x_2$, respectively.

We saw last time that system (1) was linear, and that system (2) was not.

(3): if q_1, q_2 are behaviors, then $q_1(n) = 3q_1(n-1)$ and $q_2(n) = 3q_2(n-1)$

If $q_3 = a_1 q_1 + a_2 q_2$, then $q_3(n) = a_1 q_1(n) + a_2 q_2(n) = 3a_1 q_1(n-1) + 3a_2 q_2(n-1) = 3q_3(n-1) \Rightarrow q_3$ is also a behavior $\Rightarrow$ LINEAR

b) Time-Invariance. A system S is defined to be *time-invariant* if delaying the input results in only an identical delay in the output. In other words, suppose that

- y is the response of S to x;

- y' and x' are shifted versions of y and x, respectively, with shift n_0:

$$y'(n) = y(n-n_0);$$
$$x'(n) = x(n-n_0);$$

- y_1 is the response of S to x'.

Then system S is defined to be time-invariant if, for any x and any shift n_0,

$$y_1 = y'.$$

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              ------------                   ---------------
x            |            |       y         |               |    y'
------->|  system S  |--------------->| delay by n0  |------->
             |____________|                 |_______________|


              ----------------              ------------
x            |                |     x'      |            |    y1
------->| delay by n0  |--------------->|  system S  |------->
             |________________|             |____________|


TI if y1=y' for ANY input x and any shift n0.
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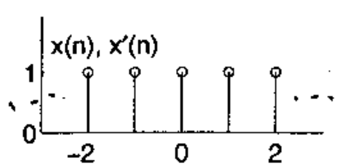
So, the concept is very simple: if you shift the input, the output is shifted by the same amount. It is like going to college: you start in 1997, you graduate in 2001; you start one year later, you graduate one year later, and so on. In time-varying systems, output changes drastically is you shift the input. An example of a time-varying system is EE 438 staff. If you turned in your first homework before 5pm on Friday, August 25, then you would get it back, graded, next Friday. However, if you shift the day you turn in your homework by, let us say, one day beyond the deadline, it is not true that our response will be the same but shifted by one day.

Example 1. Is system (1) time-invariant?

Let $x(n) = 1$:



Then, for $n_0 = 1$,

$$\begin{aligned} y'(1) &= y(1-n_0) = y(1-1) = y(0) = \sum_{k=0}^{0} x(k) = x(0) = 1; \\ x'(n) &= x(n-n_0) = x(n-1) = 1, \text{ for all } n; \\ y_1(1) &= \sum_{k=0}^{k=1} x'(k) = x'(0) + x'(1) = 1 + 1 = 2. \end{aligned}$$

Thus, $y'(1) \neq y_1(1)$, which means that the system is time-varying.

Example 2. Is system (2) time-invariant?

$$\begin{aligned} y'(n) &= y(n-n_0) = 2x(n-n_0) + 3, \text{ for all } n; \\ x'(n) &= x(n-n_0), \text{ for all } n; \\ y_1(n) &= 2x'(n-n_0) + 3 = 2x(n-n_0) + 3, \text{ for all } n. \end{aligned}$$

Therefore, $y'(n) = y_1(n)$ for all n, which proves that the system is time-invariant.

Let me emphasize once again this "for any x and any shift n_0" in the definition of time-invariance. This means, that, in order to prove that a system is time-invariant, we would have to prove it for all possible input signals, and all possible time shifts, as we did in example 2. On the other hand, in order to prove that a system is not time-invariant, we only need to come up with one specific counter-example, as we did in example 1.

Behavioral view: a system is TI if any delay of a behavior is another behavior.

If time n does not explicitly enter into the def, it's TI.
– rule of thumb

Ex. 3 Replace n with $n-n_0$: Let $p(n) = q(n-n_0)$

$p(n) = q(n-n_0) = 3q(n-n_0-1) = 3p(n-1)$

[$q(n) = 3n\,q(n-1)$ NOT TI:

~~$q(n-n_0) = 3(n-n_0)\,q(n-n_0-1)$~~ ⇒

Let $p(n) = q(n-n_0)$

$p(n) = q(n-n_0) = 3(n-n_0)\,q(n-n_0-1)$

$\neq 3n\,p(n-1) = 3n\,q(n-n_0-1)$]

• ~~For system properties, see Example 2~~

We will come back later to discussing other important properties of systems. For now, we will take a closer look at linear time-invariant systems, and develop a method to compute the output of an LTI system, given its input. Specifically, we will see that the output is the convolution of the input with the impulse response.

1.2.3. Impulse Response and Convolution.

Plan:

1. Write the input signal as a linear combination (weighted sum) of shifted unit impulse signals.
2. Use linearity to write the response as the sum of responses to shifted impulses.
3. Use time-invariance to find the response to a shifted impulse.

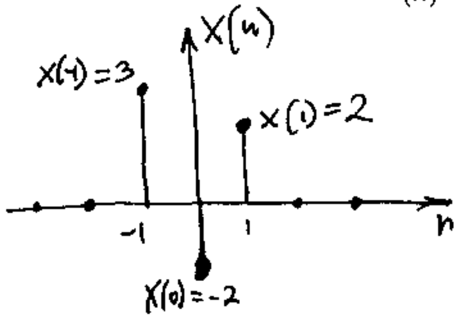
> 1. $x = \sum$ weighted, shifted impulses.
> 2. Linearity $\Rightarrow y = \sum$ weighted responses to shifted impulses.
> 3. Time-invariance $\Rightarrow$ response to $\delta(n-k)$ is $h(n-k)$, where $h(n)$ is the unit impulse response.

~~Denote by δ_k the unit impulse function, shifted by k:~~

~~$$\delta_k(n) = \delta(n-k)$$~~

Let us plot the signal

$$x(n) = 3\delta(n+1) - 2\delta(n) + 2\delta(n-1) =$$



$$\vdots$$
$$= [x(-2)\,\delta(n-(-2))]$$
$$+ x(-1)\,\delta(n-(-1))$$
$$+ x(0)\,\delta(n-0)$$
$$+ x(1)\,\delta(n-1)$$
$$[+ x(2)\,\delta(n-2)]$$
$$\vdots$$
$$= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$\xrightarrow{\text{linear system}}$

$$y(n) =$$
$$\vdots$$
$$= x(-2)\,h_{-2}(n)$$
$$+ x(-1)\,h_{-1}(n)$$
$$+ x(0)\,h_0(n)$$
$$+ x(1)\,h_1(n)$$
$$+ x(2)\,h_2(n)$$
$$\vdots$$
$$= \sum_{k=-\infty}^{\infty} x(k)h_k(n),$$

where h_k is the response to $\delta(n-k)$.

Thus, the response of a linear system S to x is:

$$y = S[x] = S\left[\sum_{k=-\infty}^{\infty} x(k)\delta_k\right] \overset{\text{linearity}}{=} \sum_{k=-\infty}^{\infty} x(k)S[\delta_k].$$

Denote the response of S to δ_k by h_k:

$$h_k = S[\delta_k].$$

Then

$$y = \sum_{k=-\infty}^{\infty} x(k)h_k$$

Evaluated at any time n, this is:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h_k(n).$$

If system S, in addition to being linear, is time-invariant, then

$$h_k(n) = h(n-k),$$

where h is the response to the impulse δ. Then we have:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$ – DT convolution

I.e., the output of a discrete-time LTI system is the convolution of the input and the impulse response.