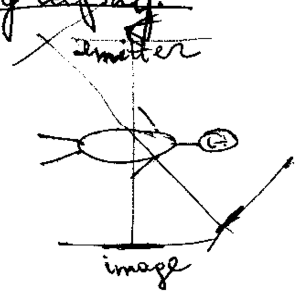
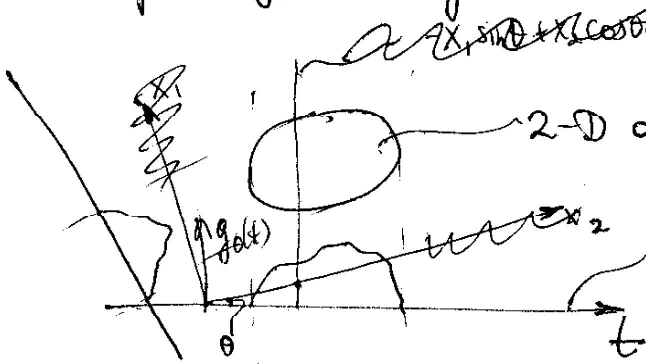
3.5 Tomography



- Shine particles through an object
- Obtain several 2-D images, for several angles.

How to reconstruct the 3-D object from the 2-D images?

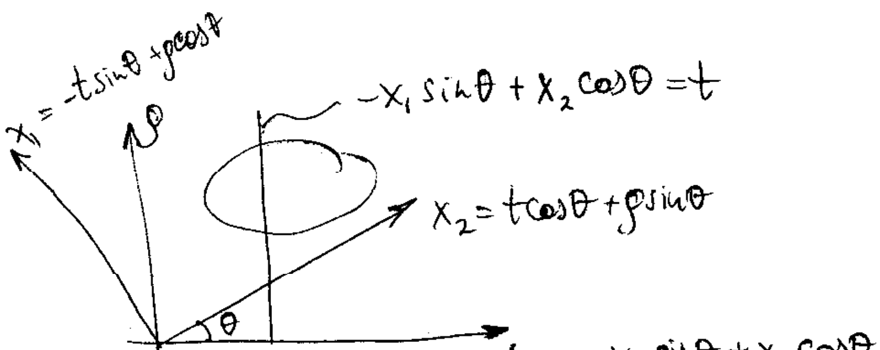
For simplicity, let's go one dimension down:



2-D object $f(x_1, x_2)$

its tomographic projection $g_\theta(t)$ which turns out to be the line integral of f over the line at location t and angle θ:

$$g_\theta(t) = \int_{-\infty}^{\infty} f(-t\sin\theta + \rho\cos\theta,\ t\cos\theta + \rho\sin\theta)\, d\rho$$



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \rho \\ t \end{pmatrix}$$

$$\begin{pmatrix} \rho \\ t \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

What is $G_\theta(\omega)$?

$$G_\theta(\omega) = \int_{-\infty}^{\infty} g_\theta(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(-t\sin\theta + \rho\cos\theta,\ t\cos\theta + \rho\sin\theta)\, d\rho \right] e^{-j\omega t} dt$$

$$\underset{d\rho\, dt = dx_1 dx_2}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)\, e^{-j\omega(-x_1\sin\theta + x_2\cos\theta)} dx_1 dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)\, e^{-j(\omega\sin\theta)x_1 - j(\omega\cos\theta)x_2} dx_1 dx_2$$

$$= \underbrace{F(-\omega\sin\theta, \omega\cos\theta)}_{\text{2-D cont-space F.T. of } f}$$

So, if we know $G_\theta(\omega)$ for all ω and all $\theta \in [0, 2\pi)$, we know $F(\omega_1, \omega_2)$ for all $\omega_1, \omega_2 \Rightarrow$ can reconstruct $f(x_1, x_2)$ with the inverse F.T.:

$$f(x_1, x_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega_1, \omega_2)\, e^{j\omega_1 x_1 + j\omega_2 x_2} d\omega_1 d\omega_2$$

polar coord. $\begin{cases} -\omega\sin\theta = \omega_1 \\ \omega\cos\theta = \omega_2 \end{cases}$

$$= \frac{1}{4\pi^2} \int_0^{2\pi}\int_{-\infty}^{\infty} \underbrace{F(-\omega\sin\theta, \omega\cos\theta)}_{G_\theta(\omega)}\, e^{-j\omega\sin\theta x_1 + j\omega\cos\theta x_2}\, \omega\, d\omega\, d\theta$$

$$= \frac{1}{4\pi^2} \int_0^{2\pi}\int_{-\infty}^{\infty} G_\theta(\omega)\, e^{-j\omega\sin\theta x_1 + j\omega\cos\theta x_2}\, \omega\, d\omega\, d\theta$$

$$g_\theta(t) \longrightarrow \boxed{\text{1-D F.T.}} \xrightarrow{G_\theta(\omega)} \boxed{\text{2-D inverse F.T.}} \xrightarrow{f(x_1, x_2)}$$

In applications, one is typically faced with <u>noisy</u> measurements of the tomographic projections $g_\theta(t)$ for a finite number of angles θ and a finite number of values of t. Recovering a reasonable approximation of $f(x_1, x_2)$ from this data is a very challenging problem.

4. Overview.

We've devoted a lot of time to considering the following picture:

$$\text{input} \longrightarrow \boxed{\text{system}} \longrightarrow \text{output}$$

E.g.,

speech	AR model estimator	AR coefficients
noisy image	median filter	de-noised image
transmitted message	noisy channel	received message

So, there are many kinds of signals and many systems that process these signals. We therefore need a framework, or a unifying language, for talking about signals and systems, so that we wouldn't have to derive everything from skratch every time we have a new problem.

We found that, in many useful practical situations, it is convenient to model signals as vectors in a vector space.

<u>Def.</u> A vector space V over a scalar field $\mathcal{F}$ (typically, $\mathcal{F}=\mathbb{C}$ or $\mathcal{F}=\mathbb{R}$) is a set for which

$$\underline{v}_1 + \underline{v}_2 \in V \quad \text{for any } \underline{v}_1, \underline{v}_2 \in V$$
$$a\underline{v} \in V \quad \text{for any } a \in \mathcal{F}, \underline{v} \in V.$$

Examples

(a) The set of all complex-valued N-point sequences ~~signals~~, $\boxed{\mathbb{C}^N}$, is a vector space over $\mathbb{C}$.

(sequence $x(n)$, $0 \le n \le N-1$, is identified with

$$\underline{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N) \end{pmatrix}$$
).

(b) The set of all complex-valued, infinite-duration, finite-energy sequences, ℓ^2, is a vector space over $\mathbb{C}$.

$$\underline{x} = \begin{pmatrix} \vdots \\ x(-1) \\ x(0) \\ x(1) \\ \vdots \end{pmatrix}$$ - infinite-dimensional space

~~(HW 2, Prob 2: if x_1, x_2 have finite energy, then so does $x_1 + x_2$.)~~

(c) The set of all complex-valued finite-energy continuous-time signals defined on a finite interval ($x(t)$ for $a \le t \le b$), is a vector space over $\mathbb{C}$, called $L^2([a,b])$.

(d) The set of all zero-mean finite-variance real-valued random variables is a vector space over $\mathbb{R}$.

An important class of systems to be analyzed are linear systems.

Def. A linear system is a mapping (or an operator) S between two vector spaces over the same field $\mathbb{F}$, V and W, such that

$$S(a_1\underline{v}_1 + a_2\underline{v}_2) = a_1 S(\underline{v}_1) + a_2 S(\underline{v}_2)$$

Therefore, if we could represent every vector of V as a linear combination of several vectors,

$$\sum_{k=-\infty}^{\infty} a_k g_k ,$$

then a linear system would be completely characterized by the responses to these vectors:

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$$\sum_{k=-\infty}^{\infty} a_k g_k \longrightarrow \boxed{S,\ \text{linear}} \longrightarrow \sum_{k=-\infty}^{\infty} a_k S(g_k)$$

$\Rightarrow$ system is completely characterized by $S(g_k)$, $-\infty < k < \infty$ — i.e., we can write the response to any other vector $\underline{v} \in V$ in terms of $S(g_k)$'s.

Thus, looking at bases for vector spaces is important for analyzing linear systems.