1.2. Systems.

(1) $y(n)=\begin{cases}\sum_{n=0}^{n} x(k) & \text{for } n\geq 0\\ 0 & \text{for } n<0\end{cases}$ linear, time varying (= not TI)

(2) $y(n)=2x(n)+3$ for all n nonlinear, TI

(3) $q(n)=3q(n-1)$ for all n LTI

1.2.3 Impulse Response and Convolution.

For an LTI system, with input x and output y,

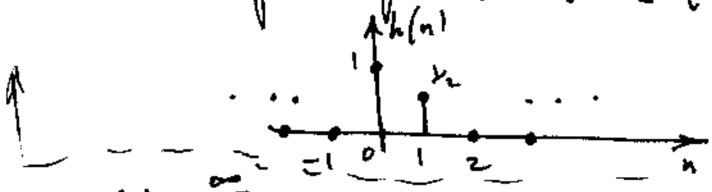
$$\underset{\text{output}}{y(n)} = \sum_{k=-\infty}^{\infty} \underset{\text{input}}{x(k)}\, \underset{\text{impulse resp.}}{h(n-k)}$$

Notation: $y(n) = x*h(n)$

Example 1. $y(n)=x(n)+\frac{1}{2}x(n-1)$

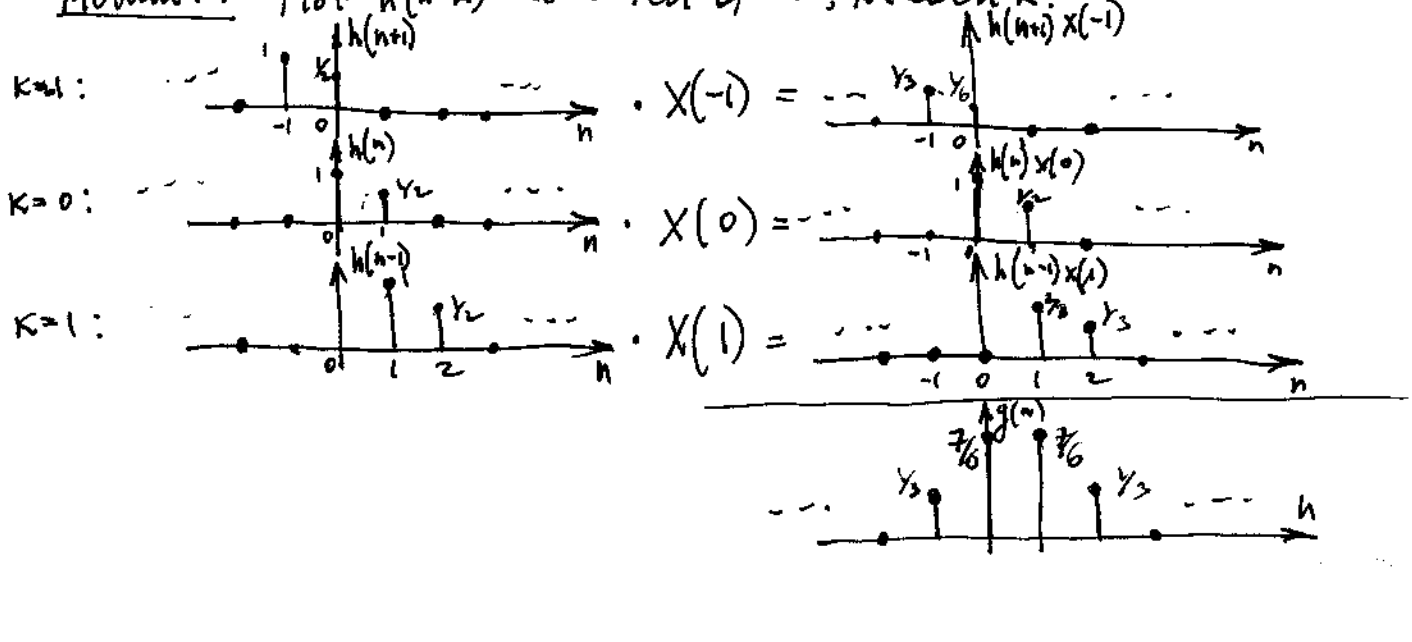
What is the response to $x(n)=\begin{cases}\frac{1}{3}, & n=-1\\ 1, & n=0\\ \frac{2}{3}, & n=1\\ 0, & \text{otherwise}\end{cases}$

Solution. Impulse resp.: $h(n)=\delta(n)+\frac{1}{2}\delta(n-1)=\begin{cases}1, & n=0\\ \frac{1}{2}, & n=1\\ 0, & \text{otherwise}\end{cases}$



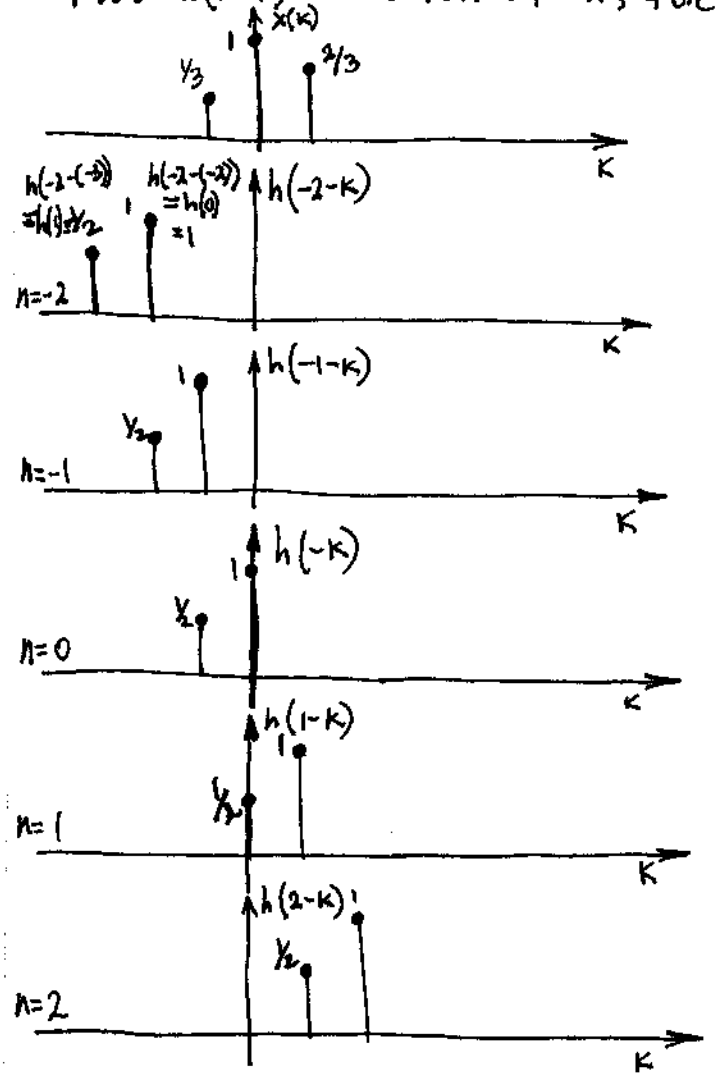
$$y(n)=\sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(-1)h(n+1)+x(0)h(n)+x(1)h(n-1)$$

Method 1: Plot $h(n-k)$ as a fcn of n, for each k.



Method 2. (Less intuitive, but more convenient).

Plot $h(n-k)$ as a fcn of k, for each n



$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)\, h(-2-k) = 0$$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k) = \frac{1}{3}\cdot 1 = \frac{1}{3}$$

$$y(0) = \frac{1}{3}\cdot\frac{1}{2} + 1\cdot 1 = \frac{7}{6}$$

$$y(1) = 1\cdot\frac{1}{2} + \frac{2}{3}\cdot 1 = \frac{7}{6}$$

$$y(2) = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}.$$

All other $y(n)$'s are zeros.

(1) flip h

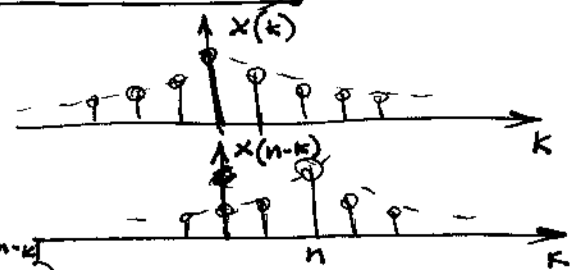
(2) shift h by n

(3) multiply $x(k)$ by $h(n-k)$, for each k

(4) Sum the products for all k: $\sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = y(n)$.

Example 2 (HW1 Prob 3)

$x(n) = 2^{-|n|}$



$y(n) = x * x(n) = \sum_{k=-\infty}^{\infty} 2^{-|k|} 2^{-|n-k|}$

use pictures as a guide, but need to calculate the sums

Case 1, n≥0 For $-\infty < k \le 0$, $|k| = -k$, $|n-k| = n-k$

for $1 \le k \le n$, $|k| = k$, $|n-k| = n-k$

for $k > n$, $|k| = k$, $|n-k| = k-n$

$$= \sum_{k=-\infty}^{0} 2^{k} 2^{k-n} + \sum_{k=1}^{n} 2^{-k} 2^{k-n} + \sum_{k=n+1}^{\infty} 2^{-k} 2^{-k+n} = \dots$$

- No sense in my doing lots of examples: need to work through them
- A very good candidate for an exam problem.
- The mechanics of this calculation don't have any grand ideas behind them – but very important.

1.2.2. Properties of Systems (continued).

c) Causality. A system is causal if the output at any time does not depend on the future values of the input:

$y(n)$ does not depend on $x(k)$ for $k > n$, for any input x and any time n.

Equivalent to:

If $x_1(n) = x_2(n)$ for $n \le n_0$, then $y_1(n) = y_2(n)$ for $n \le n_0$

Ex. (1) & (2) are causal.

We will not need behavioral definition of causality.

Causality for LTI systems.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$ – all terms involving $x(k)$ for $k > 0$, have to be zero, for any input x

therefore, $h(n-k) = 0$ for $k > n$

This is equiv. to $h(m) = 0$ for $m < 0$, where $m = n-k$.

So, an LTI system is causal iff its impulse response satisfies $h(m) = 0$ for $m < 0$

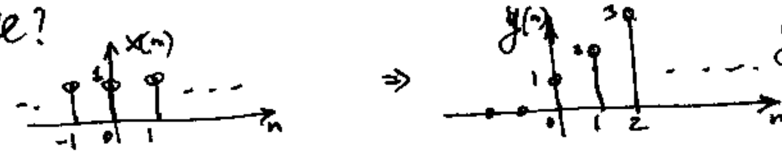
d) **BIBO Stability.** A system is said to be bounded-input-bounded-output (BIBO) stable if every bounded input x produces a bounded output y:

$$\mathcal{M}(x) \text{ is finite} \Rightarrow \mathcal{M}(y) \text{ is finite},$$

where $\mathcal{M}$ denotes the magnitude of a signal: $\mathcal{M}(x) = \max\limits_{-\infty<n<\infty} |x(n)|$.

Ex 1. Is (1) stable?

Let $x(n)=1$:



$$y(n) = \begin{cases} \sum_{k=0}^{n} 1 = n+1 & \text{for } n \geq 0 \\ 0 & \text{for } n<0 \end{cases}$$

$$\max_{-\infty<n<\infty} |y(n)| \geq \max_{0\leq n<\infty} y(n) = \max_{0\leq n<\infty} (n+1) = \infty$$

Bounded x produced an unbounded y.
Therefore, the system is not BIBO stable.

Ex 2. Is (2) stable?

$$|y(n)| = |2x(n)+3| \leq 2|x(n)|+3 \quad \text{, for all } n$$

$$\max_{-\infty<n<\infty} |y(n)| \leq 2 \max_{-\infty<n<\infty} |x(n)| + 3 = 2\mathcal{M}(x)+3.$$

So, if x is bounded (i.e., $\mathcal{M}(x)$ is finite), then y is also bounded.

BIBO Stability for LTI Systems. An LTI system is BIBO stable if and only if its impulse response is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |h(n)| \text{ is finite.} \qquad (1)$$

Proof. $\Rightarrow$: Suppose an LTI system with impulse resp h is BIBO stable.

Note that $y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$

Consider the following input signal:

$$x(k) = \begin{cases} 1, & h(-k) \geq 0 \\ -1, & h(-k) < 0 \end{cases}$$

$$\Downarrow$$

$$y(0) = \sum_{k=-\infty}^{\infty} |h(k)|.$$

Since x is bounded, and since our system is, by assumption, BIBO stable, y must be bounded, and, in particular, $y(0)$ is a finite number. This implies (1).

$\Leftarrow$: Now suppose that the impulse response is summable,

$$\sum_{k=-\infty}^{\infty} |h(k)| = L$$

and let's prove that the LTI system is then BIBO stable.

If the input is bounded, i.e., $\mathcal{M}(x)$ is finite, then we have:

$$
\begin{aligned}
y(n) &= \left| \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \right| \\
&\leq \sum_{k=-\infty}^{\infty} |x(k)| \cdot |h(n-k)| \\
&\leq \sum_{k=-\infty}^{\infty} \mathcal{M}(x)\, |h(n-k)| \\
&= \mathcal{M}(x) \sum_{k=-\infty}^{\infty} |h(n-k)| \\
&= \mathcal{M}(x)\, L .
\end{aligned}
$$

1.3 Frequency Analysis 1.3.1. - Handout

1.3.2 Some Linear Algebra.