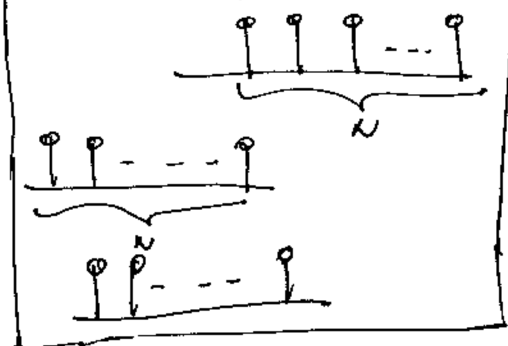
1.3 Frequency Analysis. ~~LEC 6 F01~~

1.3.1 Review of Complex Numbers. → handout

1.3.2 ~~Basic Linear Algebra Fourier~~ DTFT and Fourier se

Motivation. To derive conv. formula, had to represent the input signal as a sum of shifted impulses. Conv. sum is complicated – $O(N^2)$

$x * h(n)$, where both x, h have duration N:



· flip
· slide
· multiply

\# of arithmetic operations:

1st time 1 multiplication
2nd time 2 "
3
⋮
N
N−1
⋮
3
2
1

N^2 multiplications + some additions

$\boxed{N=1000 \Rightarrow 0.1 \text{ sec}}$
on Sun Ultra 10 in Matlab 5

1000×1000 images ⇒ ~1,000,000 longer
~ 100,000 s = 27–28 hours

What if

$g_k(n) \to$ LTI system $\to \alpha_k g_k(n)$

and

$x(n) = \sum_{k=1}^{N} a_k g_k(n)$ → LTI system → $y(n) = \sum_{k=1}^{N} a_k \alpha_k g_k(n)$

FFT (lab 6)   IFFT

⇒ Just N multiplications to go from a_k's to $a_k\alpha_k$'s.

(Note, however, that also need to go from x to its coefficients, and then back from those coefficients to y. That where the Fast Fourier Transform algorithm comes in. You'll see it in a few weeks, in Lab 6.)

Try complex exponentials.

$$\boxed{\text{Try } g(n) = e^{j\omega n}}$$

What's the response of an LTI system with impulse response $h(n)$?

Use the convolution formula:

$$\text{Response } y(n) = \sum_{m=-\infty}^{\infty} h(m) g(n-m) = \sum_{m=-\infty}^{\infty} h(m) e^{j\omega(n-m)}$$

$$= \boxed{e^{j\omega n} \underbrace{\sum h(m) e^{-j\omega m}}_{\text{call this } H(e^{j\omega})}} = g(n) H(e^{j\omega})$$

This depends on

- the system
- frequency ω,

but is indep. of n.

It is called the FREQUENCY RESPONSE of the system, and is the DTFT of the impulse response

~~prev. picture~~

$$g(n) = e^{j\omega n} \longrightarrow \boxed{\text{LTI system}} \longrightarrow y(n) = H(e^{j\omega}) e^{j\omega n}$$

Let $g_k(n) = e^{j\frac{2\pi k}{N} n}$

$$\underbrace{x(n) = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi k}{N} n}}_{\text{DT Fourier Series.}} \longrightarrow \boxed{\text{LTI system}} \longrightarrow y(n) = \sum_{k=0}^{N-1} a_k \underbrace{H\left(e^{j\frac{2\pi k}{N} n}\right)}_{\text{DTFT of } h(n)} e^{j\frac{2\pi k}{N} n}$$

CT versions: CTFS and CTFT - need them to analyze sampling

In other words, LTI systems treat signals as though the signals are composed of different frequencies. The system modifies each frequency independently of all other frequencies. Therefore, in analyzing LTI systems, it is very convenient to have spectral representations of signals - that is, decompositions of signals into their frequency components.

$$\boxed{x(n) = \sum_{k=1}^{N} a_k g_k(n)}$$

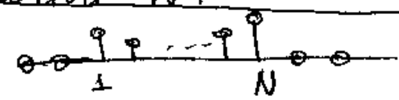
g_k: shifted impulses
complex exponentials

There is a very nice intuitive framework for looking at ~~[illegible]~~ such representations. It's called linear algebra. By using it, you get a lot of geometric insight into the above formulas which now may seem strange and complicated.

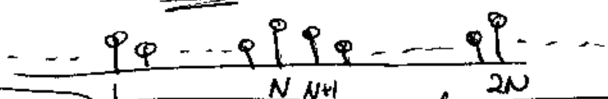
1.5.3. Basic Linear Algebra

Consider DT fcns defined only on the interval from $n=1$ to $n=N$, or periodic fcns w/ period N:

$s(n)$, $n = 1, \dots, N$



OR



In either case, the function can be thought of as an N-dim vector:

$$\boxed{\underline{s} = \begin{pmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{pmatrix}}$$

~~Some linear algebra~~

Lec 6

Announce: HW/lab questions

HW3 posted

7

1

Consider DT fcns defined only on a finite interval from $n=1$ to $n=N$:

$$\boxed{s(n), \quad n=1,\ldots,N.}$$

$$\boxed{\begin{array}{l}(z_1+z_2)^* = z_1^*+z_2^* \\ (z_1 z_2)^* = z_1^* z_2^*\end{array}}$$

Each such function can be thought of as an N-dim vector:

$$\boxed{\underline{s} = \begin{pmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{pmatrix}}$$

a) Inner Products and Orthogonality.

Definition. The inner product of two complex-valued vectors $\underline{s} = \begin{pmatrix} s(1) \\ \vdots \\ s(N) \end{pmatrix}$ and $\underline{g} \begin{pmatrix} g(1) \\ \vdots \\ g(N) \end{pmatrix}$ is

$$\boxed{\langle \underline{s}, \underline{g} \rangle = \sum_{n=1}^{N} s(n) \big(g(n)\big)^*}$$

(Note that, when both $\underline{s}$ and $\underline{g}$ are real-valued, the complex conjugate can be dropped:

$$\langle \underline{s}, \underline{g} \rangle = \sum_{n=1}^{N} s(n)\, g(n) = \underline{s}^T \underline{g}.)$$

~~Example. When N=2, and s, g are real-valued, we are dealing with vectors in a plane.~~

~~Let $\begin{pmatrix} s(1) \\ s(2) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\begin{pmatrix} g(1) \\ g(2) \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$~~

Two vectors $\underline{s}, \underline{g}$ are defined to be orthogonal if their inner product is zero:

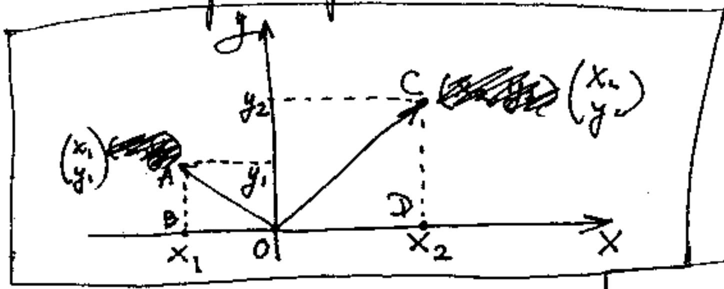
$$\boxed{\underline{s} \perp \underline{g}: \quad \langle \underline{s}, \underline{g} \rangle = 0.}$$

(Denoted by $\underline{s} \perp \underline{g}$.)

Ex. When $N=2$ and $\underline{s}, \underline{g}$ are real-valued, they are vectors in a plane.

Denote $\boxed{\begin{pmatrix} s(1) \\ s(2) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} g(1) \\ g(2) \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}}$

(Note, by the way, that this plane has nothing to do with the complex plane we considered ~~before~~ earlier.)



Then their inner product is $\langle \underline{s}, \underline{g} \rangle = \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = x_1 x_2 + y_1 y_2$.

Suppose that the inner product is zero:

then $\boxed{\begin{aligned} x_1 x_2 + y_1 y_2 &= 0 \\ -\frac{x_1}{y_1} &= \frac{y_2}{x_2} \end{aligned}}$ $\Rightarrow$ triangles OBA and CDO are similar $\Rightarrow$ exercise

$\Rightarrow \angle AOB = \angle OCD$

$\angle BAO = \angle DOC$

$\boxed{\Rightarrow \angle COA = 90^\circ}$ in a plane.

A similar argument applies if these two vectors are oriented differently w.r.t. the axes.

So, "two vectors are orthogonal"

means

"they form a 90° angle"

So, the reason for the introduction of linear algebra into this course is to be able to think of signals as **vectors**, in which case we can draw pictures like this, to help ~~[illegible]~~ develop our intuition about signal processing problems.

Properties of inner products.

1) $\langle \underline{g}, \underline{s} \rangle = \langle \underline{s}, \underline{g} \rangle^*$ (if real-valued, then $\langle \underline{g}, \underline{s}\rangle = \langle \underline{s}, \underline{g}\rangle$).

$$\left[\sum_{n=1}^{N} s(n)(g(n))^*\right]^* =$$

$$= \sum_{n=1}^{N} \left[s(n)(g(n))^*\right]^* = \sum_{n=1}^{N} (s(n))^* ((g(n))^*)^* =$$

$$= \sum_{n=1}^{N} (s(n))^* g(n) = \langle \underline{g}, \underline{s} \rangle$$

2)

$$\langle a_1 \underline{s}_1 + a_2 \underline{s}_2, \underline{g} \rangle = a_1 \langle \underline{s}_1, \underline{g} \rangle + a_2 \langle \underline{s}_2, \underline{g} \rangle$$ – exercise

$$\langle \underline{s}, a_1 \underline{g}_1 + a_2 \underline{g}_2 \rangle = \langle a_1 \underline{g}_1 + a_2 \underline{g}_2, \underline{s} \rangle^* =$$

$$= \left[a_1 \langle \underline{g}_1, \underline{s}\rangle + a_2 \langle \underline{g}_2, \underline{s} \rangle\right]^* =$$

$$= a_1^* \langle \underline{g}_1, \underline{s} \rangle^* + a_2^* \langle \underline{g}_2, \underline{s} \rangle^* =$$

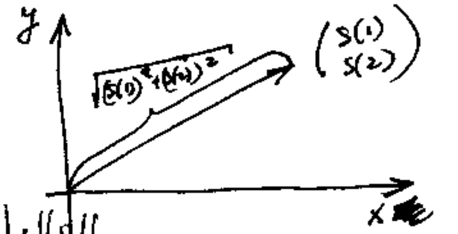
$$= a_1^* \langle \underline{s}, \underline{g}_1 \rangle + a_2^* \langle \underline{s}, \underline{g}_2 \rangle.$$

3) $\langle \underline{s}, \underline{s} \rangle$ is always a real number, and is the energy of $\underline{s}$:

$$\langle \underline{s}, \underline{s} \rangle = \sum_{n=1}^{N} s(n)(s(n))^* = \sum_{n=1}^{N} |s(n)|^2$$

(Note that, when $\underline{s}$ is real-valued, $|s(n)|^2 = (s(n))^2$, and so this definition reduces to our previous definition of energy for real-valued signals.)

4) $\sqrt{\langle \underline{s}, \underline{s} \rangle}$ is called the (l_2) norm of $\underline{s}$, and is denoted by $\|\underline{s}\|_2$. Note that, when $\underline{s} = \begin{pmatrix} s(1) \\ s(2) \end{pmatrix} \in \mathbb{R}^2$,



5) $\|a\, s\| = |a| \cdot \|s\|$

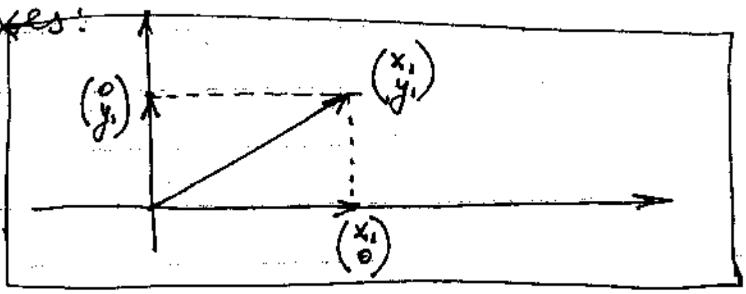
Need in HW3 Prob 1: $\|a s\| = |a| \cdot \|s\|$ — exercise
[didn't do]

Let me repeat again that our goal is to view signals as geometric objects, for which the notions of length, angle, orthogonality are well-defined. This way, a lot of signal processing ideas which would otherwise be complicated, are reduced to drawing simple pictures.

6) Orthogonal Projections.

Need inner products & orthogon. in order to be able to generalize the notions of coordinate systems to N dimensions.

In a plane, the coordinates of a vector are given by the projections of the vector onto the coordinate axes:



We will see that the coordinates of a signal in a Fourier basis – that is, the Fourier series coefficients – can also be computed from the projections of the signal onto the individual sinusoids.

So, what is the orthogonal projection of a vector $\underline{s} = \begin{pmatrix} s(1) \\ \vdots \\ s(N) \end{pmatrix}$ onto a vector $g = \begin{pmatrix} g(1) \\ \vdots \\ g(N) \end{pmatrix}$?