1.3.3 Basic Linear Algebra (continued):

$s(n)=\sum_k a_k g_k(n)$ (Lec 7 F01, wed 9/5) 1

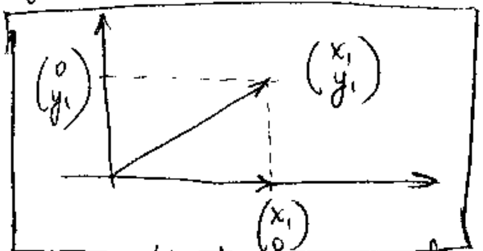
a) Inner Products and Orthogonality:

We need ~~the~~ inner products and orthogonality to be able to generalize notions related to angles to N dimensions. We need norms to be able to generalize the notion of distance.

Now let's look at how to calculate the coordinates of a vector in an arbitrary coordinate system in N dimensions.

b) Orthogonal Projections

In a plane, the coordinates of a vector are given by the projections of the vector onto the coordinate axes



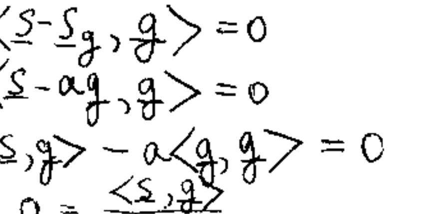
We will soon see that the coordinates of a signal in a Fourier basis - that is, the Fourier series coefficients - can also be computed from the projections of the signal onto the individual complex exponentials.

So, what is the orthogonal projection of a vector $\underline{s}=\begin{pmatrix}s(1)\\ \vdots\\ s(N)\end{pmatrix}$ onto a vector $\underline{g}=\begin{pmatrix}g(1)\\ \vdots\\ g(N)\end{pmatrix}$?

Definition. The orthogonal projection of a vector $\underline{s}=\begin{pmatrix}s(1)\\ \vdots\\ s(N)\end{pmatrix}$ onto a non-zero vector $\underline{g}=\begin{pmatrix}g(1)\\ \vdots\\ g(N)\end{pmatrix}$ is the vector $\underline{s}_g$, such that:

1) $\underline{s}_g = a\underline{g}$ for some complex number a

2) $(\underline{s}-\underline{s}_g)\perp \underline{g}$

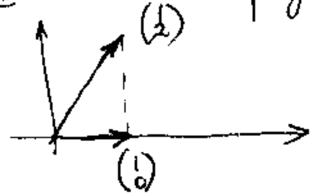


What is "a"?

$$\langle \underline{s}-\underline{s}_g, \underline{g}\rangle = 0$$
$$\langle \underline{s}-a\underline{g}, \underline{g}\rangle = 0$$
$$\langle \underline{s}, \underline{g}\rangle - a\langle \underline{g}, \underline{g}\rangle = 0$$
$$\Rightarrow a = \frac{\langle \underline{s}, \underline{g}\rangle}{\langle \underline{g}, \underline{g}\rangle}$$

Thus, $\boxed{\underline{s}_g = \frac{\langle \underline{s}, \underline{g}\rangle}{\langle \underline{g}, \underline{g}\rangle}\,\underline{g}}$

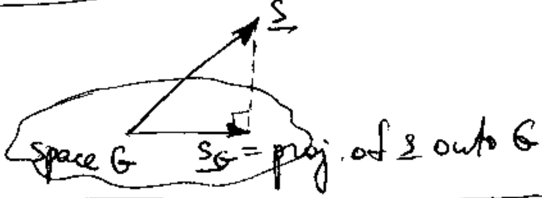
Example. What's the proj. of $\binom{1}{2}$ onto $\binom{1}{0}$?



-should be $\binom{1}{0}$.

$$\frac{\left\langle \binom{1}{2}, \binom{1}{0} \right\rangle}{\left\langle \binom{1}{0}, \binom{1}{0} \right\rangle} \binom{1}{0} = \frac{1\cdot 1 + 2\cdot 0}{1\cdot 1 + 0\cdot 0} \cdot \binom{1}{0} = \binom{1}{0}$$

How to project a vector $\underline{s}$ onto a vector subspace G?



~~Definition. The set of all N-point vectors is called $\mathbb{R}^N$. A subset G of $\mathbb{R}^N$ is called a vector subspace of $\mathbb{R}^N$ if~~

$\mathbb{C}^N$ ~~$ag \in G$ for any $g \in G$ and any number a~~

~~"is an element of"~~

~~and $g_1 + g_2 \in G$ for any $g_1, g_2 \in G$.~~

~~Example. The set~~

Definition. The set of all N-point real-valued vectors is called $\mathbb{R}^N$.
" " " " " complex- " " " $\mathbb{C}^N$.

A subset G of
$\mathbb{R}^N$ | $\mathbb{C}^N$
is called a vector subspace of
$\mathbb{R}^N$ | $\mathbb{C}^N$
if $ag \in G$ for any $g \in G$ and for any
"is an element of" $a \in \mathbb{R}$ | $a \in \mathbb{C}$
and $g_1 + g_2 \in G$ for any $g_1, g_2 \in G$

Example. The set of all vectors of the form $\begin{pmatrix}\alpha\\0\\0\end{pmatrix}$ where $\alpha \in \mathbb{C}$, is a vector subspace of $\mathbb{C}^3$ (why?). The set of all vectors of the form $\begin{pmatrix}\alpha\\1\end{pmatrix}$ is NOT a vector subspace of $\mathbb{C}^3$ (why?).

Definition. Vectors $g_1, \ldots, g_m$ are called linearly independent if
$a_1 g_1 + a_2 g_2 + \ldots + a_m g_m = 0$ implies $a_1 = a_2 = \ldots = a_m = 0$.

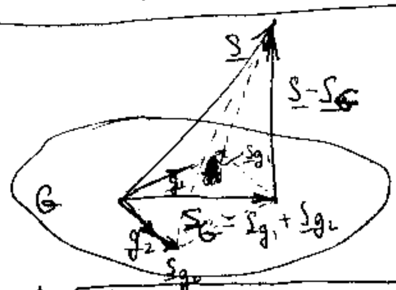
Definition. The space spanned by vectors $g_1, \ldots, g_m$ is the set of all their linear combinations – i.e., all vectors of the form $\sum_{k=1}^{m} a_k g_k$, where $a_1, \ldots, a_k$ are numbers.

Definition. If $G = \text{span}\{g_1, \dots, g_m\}$, and if $g_1, \dots g_m$ are linearly independent, then $\{g_1, \dots, g_m\}$ is said to be a basis for space G. If, in addition, $g_1, \dots, g_m$ are pairwise orthogonal, the basis is said to be an orthogonal basis.

Ex. $\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \mathbb{C}^2$, since $\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}$ for any $\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{C}^2$.
$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ are lin. ind. (exercise). Also, $\begin{pmatrix}1\\0\end{pmatrix} \perp \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow$ orth basis.

Definition. The orth. projection of $\underline{s}$ onto G is the vector $\underline{s}_G$ s.t.:

1) $\underline{s}_G \in G$
2) $(\underline{s} - \underline{s}_G) \perp G$ (i.e., $s - s_G \perp$ to any vector in G)



Suppose that $\{g_1, \dots, g_m\}$ is an orth. basis of G. Then $\underline{s}_G \in G$ means $\underline{s}_G = \sum_{k=1}^{m} a_k g_k$, for some numbers $a_1, \dots, a_m$.

How to compute the projection coefficients in terms of s and $g_1, \dots, g_m$?

"$(\underline{s}-\underline{s}_G) \perp G$" implies $(\underline{s}-\underline{s}_G) \perp g_1, (\underline{s}-\underline{s}_G) \perp g_2, \dots, (\underline{s}-\underline{s}_G) \perp g_m$:

$$\langle \underline{s} - \underline{s}_G, g_p \rangle = 0, \quad \text{for any } p = 1, \dots, m$$

$$\langle \underline{s}, g_p \rangle - \langle \underline{s}_G, g_p \rangle = 0$$

$$\langle \underline{s}, g_p \rangle - \left\langle \sum_{k=1}^{m} a_k g_k, g_p \right\rangle = 0$$

$$\langle s, g_p \rangle - \sum_{k=1}^{m} a_k \underbrace{\langle g_k, g_p \rangle}_{=0 \text{ if } k \neq p} = 0$$

$$\langle \underline{s}, g_p \rangle - a_p \langle g_p, g_p \rangle = 0$$

$$\boxed{a_p = \frac{\langle \underline{s}, g_p \rangle}{\langle g_p, g_p \rangle} \quad \text{for } p = 1, \dots, m}$$

Thus $$\boxed{\underline{s}_G = \sum_{k=1}^{m} a_k g_k = \sum_{k=1}^{m} \frac{\langle \underline{s}, g_k \rangle}{\langle g_k, g_k \rangle} g_k}$$

(• project onto the individual basis vectors
• sum the results.

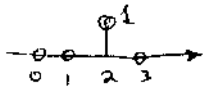
In particular, if $\underline{s} \in G$, then $\underline{s} = \underline{s}_G = \sum_{k=1}^{m} \frac{\langle s, g_k \rangle}{\langle g_k, g_k \rangle} g_k$

1.3.4 DT Fourier Series, Revisited.

Example 1. $g_k(n) = e^{\frac{j2\pi kn}{4}}$, for $n=0,1,2,3$; $k=0,1,2,3$.

(a) Prove that $g_k \perp g_p$ for $k \neq p$; find $\underbrace{\|g_k\|^2}_{=4}$. } ~~HW 4, Prob 3.~~

(b) Let $s(n)=$



. Find a_0, a_1, a_2, a_3 in:

$$s(n) = a_0 g_0(n) + a_1 g_1(n) + a_2 g_2(n) + a_3 g_3(n) \qquad (*)$$

Solution. (a) HW 4, Prob. 3. $\|g_k\|^2 = 4$.

(b) From linear algebra:

N pairwise orthogonal non-zero vectors in $\mathbb{C}^N$ form an orth. basis for $\mathbb{C}^N$

$\Rightarrow \{g_0, g_1, g_2, g_3\}$ is a basis for $\mathbb{C}^4$

$\Rightarrow$ representation $(*)$ is indeed possible, and

$$a_0 = \frac{\langle s, g_0\rangle}{\|g_0\|^2} = \frac{1}{4}$$

$$a_1 = \frac{\langle s, g_1\rangle}{\|g_1\|^2} = -\frac{1}{4}$$

$$a_2 = \frac{\langle s, g_2\rangle}{\|g_2\|^2} = \frac{1}{4}$$

$$a_3 = \frac{\langle s, g_3\rangle}{\|g_3\|} = -\frac{1}{4}$$

exercise.

Let us generalize this to N dimensions.

Example 3. Consider the following DT complex exponential fcns:

$$g_k(n) = e^{\frac{j2\pi kn}{N}}, \quad n=0,\dots,N-1; \ k=0,\dots,N-1.$$

$$g_0 = \begin{pmatrix} g_0(0) \\ g_0(1) \\ \vdots \\ g_0(N-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} g_1(0) \\ g_1(1) \\ \vdots \\ g_1(N-1) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{\frac{j2\pi}{N}} \\ \vdots \\ e^{\frac{j2\pi(N-1)}{N}} \end{pmatrix}, \dots, \quad g_{N-1} = \begin{pmatrix} g_{N-1}(0) \\ g_{N-1}(1) \\ \vdots \\ g_{N-1}(N-1) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{\frac{j2\pi(N-1)}{N}} \\ \vdots \\ e^{\frac{j2\pi(N-1)(N-1)}{N}} \end{pmatrix}$$

(a) Prove that $g_k \perp g_p$ for $k \neq p$; find $\|g_k\|^2$.

(b) Find a formula for the Fourier series coefficients $\{a_0, \dots, a_{N-1}\}$ of an N-pt complex-valued signal $s(n)$

$$s(n) = \sum_{k=0}^{N-1} a_k g_k(n)$$

Solution. (a) HW 4, Prob 3. $\|g_k\|^2 = N$.

(B) $$\boxed{a_k = \frac{\langle s, g_k\rangle}{\langle g_k, g_k\rangle} = \frac{1}{N}\langle s, g_k\rangle = \frac{1}{N}\sum_{n=0}^{N-1} s(n)\, g_k^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} s(n)\, e^{-j\frac{2\pi k n}{N}}}$$ — "analysis formula": analyzing the signal $s(n)$ into its frequency components

$$\boxed{s(n) = \sum_{k=0}^{N-1} a_k g_k(n) = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi k n}{N}}}$$ — "synthesis formula" synthesizing the signal $s(n)$ from its frequency components.

Note: the complex exponentials could be ~~given~~ defined for a slightly different set of indexes (as in HW 4), or with a different normalization. ~~As a result~~, The Fourier series formulas would then look somewhat different. However, there is no need to memorize the formulas. As long as you remember their meaning you can re-derive them quite easily. And the meaning is that we are computing the coefficients in the decomposition of a signal into an orthogonal basis.

Summary: how to write $\underline{s} = \sum_{k=0}^{N-1} a_k g_k$, where $g_0, \ldots, g_{N-1}$ are nonzero, pairwise orthogonal?

Answer: $a_k g_k$ = projection of $\underline{s}$ onto g_k,

$$a_k = \frac{\langle s, g_k\rangle}{\langle g_k, g_k\rangle}.$$

The importance of this is that we've reduced rather complicated manipulations with complicated signals to a very intuitive geometric interpretation: namely, calculating the coordinates of a vector. ~~Also~~
Another important consequence is that this is a general framework, which we can (and will) apply to other kinds of orthogonal bases.

1.3.5. CT Fourier Series, reviewed.

The notion of orthogonal bases and projections can be extended to spaces of CT signals. Determining whether a series representation converges (and what it converges to) is much more complicated than for finite-duration DT signals. We therefore will restrict ourselves to just one example – CT Fourier series – which you've already seen in 301 and lab 3, and which is very well understood.

Periodic CT signals, $s(t)$ and $g(t)$, with period T_0

$$\langle s, g\rangle \overset{\text{def}}{=} \int_{\tau}^{\tau+T_0} s(t)\,(g(t))^*\,dt, \text{ where } \tau \text{ is arbitrary.}$$

It turns out that, if we define the inner product this way, we can still use our magic projection formula.

$$\boxed{g_k(t) = e^{j\frac{2\pi k}{T_0}t}, \quad k = 0, \pm 1, \pm 2, \ldots}$$ – this is an ortho basis for {all T_0-periodic $s(t)$ for which $\int_\tau^{\tau+T_0} |s(t)|^2 dt < \infty$}

$$\boxed{s(t) = \sum_{k=-\infty}^{\infty} a_k g_k(t) = \sum_{k=-\infty}^{\infty} \underset{\substack{\uparrow \\ \text{what are } a_k\text{'s?}}}{a_k} e^{j\frac{2\pi k}{T_0}t}}$$