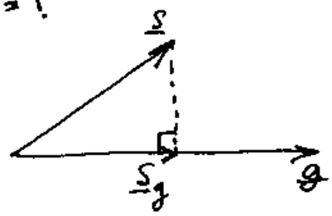
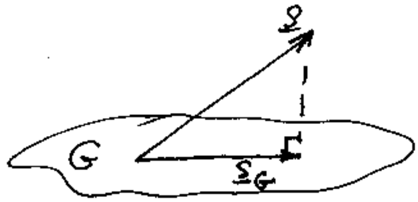
$s(n) = \sum_k a_k g_k(n)$

$a_k = ?$

$\underline{s} = \begin{pmatrix} s(1) \\ \vdots \\ s(N) \end{pmatrix}$




$$\boxed{\underline{s}_g = \frac{\langle \underline{s}, g \rangle}{\langle g, g \rangle} g}$$




$$\boxed{\underline{s}_G = \sum_{k=1}^{m} \frac{\langle \underline{s}, g_k \rangle}{\langle g_k, g_k \rangle} g_k,}$$

where $\{g_1, \ldots, g_m\}$ is an orth. basis of $G$

i.e., to compute the $k$-th coordinate, project onto the $k$-th basis vector.

In particular, if $\underline{s} \in G$, then $\underline{s} = \underline{s}_G$

Example: DT Fourier series

$$g_k(n) = e^{\frac{j2\pi kn}{N}}, \quad n = 0, \ldots, N-1; \; k = 0, \ldots, N-1.$$

$$s(n) = \sum_{k=0}^{N-1} a_k e^{\frac{j2\pi kn}{N}}$$

where $a_k = \frac{1}{N} \sum_{n=0}^{N-1} s(n) e^{-\frac{j2\pi kn}{N}}$

## 1.3.5. CT Fourier Series Reviewed.

The notions of orthogonal bases and projections can be extended to spaces of CT signals. Determining whether a series representation converges (and what it converges to) is much more complicated than for finite-duration DT signals. We therefore will restrict ourselves to just one example – CT Fourier series – which you've already seen in 301 and in Lab 3, and which is very well understood.

Consider $L_2[\tau, \tau+T_0] \stackrel{\text{def}}{=} \{$all CT $T_0$-periodic signals $s(t)$ for which $\int_\tau^{\tau+T_0} |s(t)|^2 dt < \infty \}$.

Let $\langle s, g \rangle \stackrel{\text{def}}{=} \int_\tau^{\tau+T_0} s(t)(g(t))^* dt$, where $\tau$ is arbitrary.

It turns out that then

$g_k(t) = e^{j\frac{2\pi k}{T_0} t}$, $k = 0, \pm 1, \pm 2, \ldots$ is an orth. basis for $L_2[\tau, \tau+T_0]$

We can write: $\boxed{s(t)} = \boxed{\sum_{k=-\infty}^{\infty} a_k g_k(t)} = \boxed{\sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi k}{T_0} t}}$ DNE

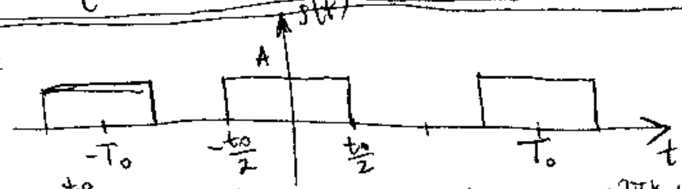
[this "=" needs careful interpretation, but we'll not go into that]

To find $a_k$'s, we can still use our magic projection formula:

$$a_k = \frac{\langle s, g_k\rangle}{\langle g_k, g_k\rangle} = \boxed{\frac{1}{T_0}\int_{\tau}^{\tau+T_0} s(t)\, e^{-j\frac{2\pi k}{T_0}t}\, dt} \text{ — DNE}$$

$$\langle g_k, g_i\rangle = \int_{\tau}^{\tau+T_0} e^{j\frac{2\pi k}{T_0}t}\, e^{-j\frac{2\pi i}{T_0}t}\, dt = \int_{\tau}^{\tau+T_0} e^{j\frac{2\pi(k-i)}{T_0}t}\, dt = \begin{cases} T_0, & k=i \\ 0, & k\neq i \end{cases}$$

Example




$$a_k = \frac{1}{T_0}\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} A\, e^{-j\frac{2\pi k}{T_0}t}\, dt = \frac{A}{T_0}\,\frac{T_0}{-j2\pi k}\, e^{-j\frac{2\pi k}{T_0}t}\Big|_{-\frac{t_0}{2}}^{\frac{t_0}{2}} =$$

$$= \frac{A}{\pi k}\cdot \underbrace{\frac{1}{2j}\left[e^{j\frac{\pi k t_0}{T_0}} - e^{-j\frac{\pi k t_0}{T_0}}\right]}_{\sin\left(\frac{\pi k t_0}{T_0}\right)} = \frac{A}{\pi k}\sin\left(\frac{\pi k t_0}{T_0}\right).$$

Trigonometric Fourier series: Notes (web), Lab 3.

## 1.3.6. Discrete-Time Fourier Transform (DTFT), Revisited.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

IDTFT: change of variables in CTFS: $T_0 = 2\pi$, $\omega = \frac{2\pi t}{T_0} = t$, $n = -k$, $x(n) = a_{-k}$

$$x(n) = \frac{1}{2\pi}\int_{\omega_0}^{\omega_0+2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$\boxed{x(n) \text{ is the CTFS coefficients of its DTFT } X(e^{j\omega})}$

So, by having that nice ~~geometric~~ framework of orthogonal projections, we ~~got three~~ easily got three very important representations, as particular cases: the DTFS, CTFS, and DTFT.

Properties.

Linearity:
DTFT of $a_1 x_1(n) + a_2 x_2(n)$ is $a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$

Delay:
DTFT of $x(n-n_0)$ is $\sum_{n=-\infty}^{\infty} x(n-n_0)\, e^{-j\omega(n-n_0)}\, e^{-j\omega n_0}$

$$\overset{m=n-n_0}{=} \underbrace{\sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m}}_{X(e^{j\omega})}\, e^{-j\omega n_0} = X(e^{j\omega})\, e^{-j\omega n_0}$$

(3) Convolution

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$Y(e^{j\omega}) = \sum_n y(n) e^{-j\omega n}$$
$$= \sum_n \left[\sum_k h(n-k)x(k)\right] e^{-j\omega n}$$
$$= \sum_k \left[\sum_n h(n-k) e^{-j\omega n}\right] x(k)$$
$$\overset{\text{property (2)}}{=} \sum_k \left[ H(e^{j\omega}) e^{-j\omega k}\right] x(k)$$
$$= H(e^{j\omega}) \underbrace{\left(\sum_k e^{-j\omega k} x(k)\right)}_{\parallel \; X(e^{j\omega})} = H(e^{j\omega}) X(e^{j\omega})$$

So, if $y(n) = h * x(n)$, then $Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$.

(4) Parseval's theorem:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$
$$\sum_{n=-\infty}^{\infty} x(n) y^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) Y^*(e^{j\omega}) d\omega$$

(5) $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x(n)$

(6) $x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega$

(7) Modulation

DTFT of $x(n)e^{j\omega_0 n}$ is $\sum_n x(n) e^{j(\omega_0 - \omega)n} = X(e^{j(\omega-\omega_0)})$

---

<u>Example 1.</u> $y(n) = \frac{1}{2}\left(x(n) + x(n-1)\right)$

Find freq. resp. $H(e^{j\omega})$. Is it low-pass, band-pass, or high-pass?

<u>Method 1.</u> Use the eigenfunction property: $x(n) = e^{j\omega n} \Rightarrow y(n) = H(e^{j\omega}) e^{j\omega n}$.

$$y(n) = \frac{1}{2}\left(e^{j\omega n} + e^{j\omega(n-1)}\right) = \underbrace{\frac{1}{2}\left(1 + e^{-j\omega}\right)}_{H(e^{j\omega})} e^{j\omega n}$$

$$H(e^{j\omega}) = \frac{1}{2} e^{-j\frac{\omega}{2}} \left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right) = e^{-j\frac{\omega}{2}} \cos\frac{\omega}{2}$$

<u>Method 2.</u> $Y(e^{j\omega}) = \text{DTFT}\left\{\frac{1}{2}\left(x(n) + x(n-1)\right)\right\}$

$$\overset{\text{linearity}}{=} \frac{1}{2}\text{DTFT}\{x(n)\} + \frac{1}{2}\text{DTFT}\{x(n-1)\}$$
$$= \frac{1}{2} X(e^{j\omega}) + \frac{1}{2} X(e^{j\omega}) e^{-j\omega}$$
$$= \frac{1}{2} X(e^{j\omega}) \left(1 + e^{-j\omega}\right)$$

$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{2}\left(1 + e^{-j\omega}\right)$ — same result

## Method 3



Impulse resp: $h(n) = \frac{1}{2}\left(\delta(n) + \delta(n-1)\right)$

$H(e^{j\omega}) = DTFT\{h(n)\}$

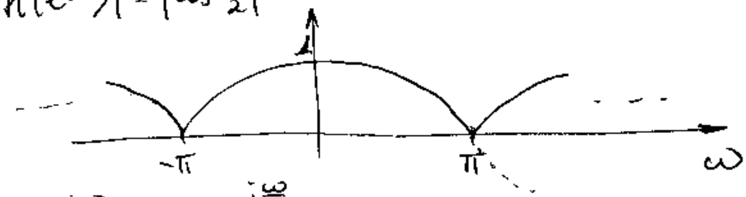
$DTFT\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n} = 1$

$DTFT\{\delta(n-1)\} = 1 \cdot e^{-j\omega \cdot 1} = e^{-j\omega}$

$H(e^{j\omega}) = \frac{1}{2}\left(1 + e^{-j\omega}\right)$

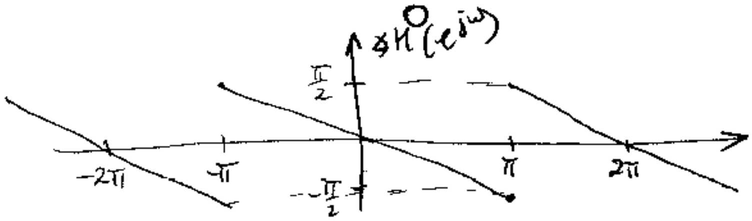
Plot $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$

$|H(e^{j\omega})| = |\cos\frac{\omega}{2}|$




$$\measuredangle H(e^{j\omega}) = \underbrace{\measuredangle e^{-j\frac{\omega}{2}}}_{-\frac{\omega}{2}} + \underbrace{\measuredangle \cos\frac{\omega}{2}}_{0}$$

$\geq 0$ for $-\frac{\pi}{2} \leq \frac{\omega}{2} \leq \frac{\pi}{2}$, i.e., $-\pi \leq \omega \leq \pi$




Note: periodic w/ period $2\pi$

low-pass

~~did~~ [exercise]

Example 2.

$y(n) = \frac{1}{2}\left(x(n) - x(n-2)\right)$

$h(n) = \frac{1}{2}\left(\delta(n) - \delta(n-2)\right)$

$$H(e^{j\omega}) = \frac{1}{2}\left(1 - e^{-2j\omega}\right) = je^{-j\omega}\underbrace{\left[\frac{1}{2j}\left(e^{j\omega} - e^{-j\omega}\right)\right]}_{\sin\omega} = je^{-j\omega}\sin\omega$$

$|H(e^{j\omega})| = |\sin\omega|$

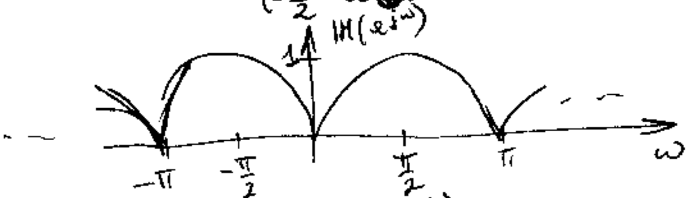
$$\angle H(e^{j\omega}) = \underbrace{\angle j}_{\frac{\pi}{2}} + \underbrace{\angle e^{-j\omega}}_{-\omega} + \angle \sin\omega$$

$\sin\omega \geq 0$ for $0 \leq \omega \leq \pi$

$\sin\omega \leq 0$ for $-\pi \leq \omega \leq 0$

$$= \begin{cases} \frac{\pi}{2} - \omega & 0 \leq \omega \leq \pi \\ -\frac{\pi}{2} - \omega & -\pi \leq \omega \leq 0 \end{cases}$$

$\leftarrow$ (convention: keep angles in the range $[-\pi, \pi]$




Bandpass