EE 438
Homework 3, due Friday, 9/6/2002.

Problem 1. Geometry of the Space $\mathbb{C}^N$ of N-point Signals.
This exercise demonstrates that N-dimensional vectors behave very similarly to 2-dimensional vectors in a plane. Recall that an N-point discrete-time complex-valued signal,

$$s(n), \qquad n = 1, \ldots, N,$$

can be identified with an N-dimensional vector $\mathbf{s} \in \mathbb{C}^N$, by recording all N values of $s(n)$ in a vector:

$$\mathbf{s} = \begin{pmatrix} s(1) \\ s(2) \\ \vdots \\ s(N) \end{pmatrix}.$$

(Note that when we write, we denote vectors by underlining them ($\underline{s}$), but in printed texts it is customary to use boldface letters ($\mathbf{s}$) to denote vectors. Another remark on notation is that a transpose of a column vector is a row vector, thus, an equivalent expression for $\mathbf{s}$ is: $\mathbf{s} = (s(1), s(2), \ldots, s(N))^T$.) The inner product of two complex-valued N-dimensional vectors is

$$\langle \mathbf{s}, \mathbf{g} \rangle = \sum_{n=1}^{N} s(n) \, (g(n))^* ,$$

and the length, or ℓ_2 norm, of a vector $\mathbf{s}$, is denoted by $\|\mathbf{s}\|$, and defined as $\|\mathbf{s}\| = \sqrt{\langle \mathbf{s}, \mathbf{s} \rangle}$.

(a) Prove the Pythagoras's theorem: the sum of energies of two orthogonal vectors is equal to the energy of their sum, i.e.,

$$\text{if } \langle \mathbf{s}, \mathbf{g} \rangle = 0, \text{ then } \|\mathbf{s}\|^2 + \|\mathbf{g}\|^2 = \|\mathbf{s} + \mathbf{g}\|^2.$$

Note that this result generalizes the following 2-D picture:

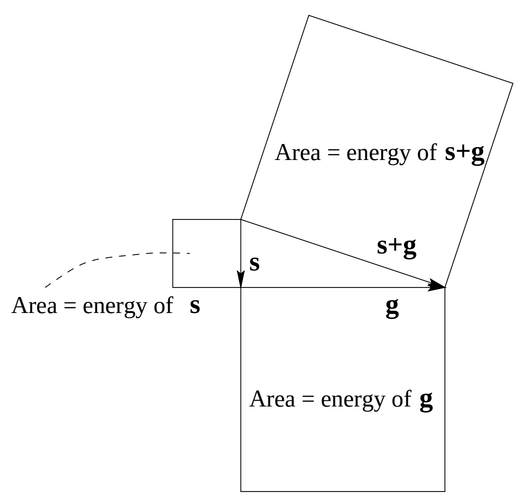

Figure 1: The sum of squares of two sides of a right triangle is equal to the square of the hypotenuse.

(**Hint.** Use the fact that $\|\mathbf{s} + \mathbf{g}\|^2 = \langle \mathbf{s} + \mathbf{g}, \mathbf{s} + \mathbf{g} \rangle$. Write this out using properties of inner products discussed in class; then use orthogonality of $\mathbf{s}$ and $\mathbf{g}$ to cancel some terms.)

(b) The projection $\mathbf{s_g}$ of a vector $\mathbf{s}$ onto another vector $\mathbf{g}$ is:

$$\mathbf{s_g} = \frac{\langle \mathbf{s}, \mathbf{g} \rangle}{\|\mathbf{g}\|^2} \mathbf{g}.$$

Show that $\mathbf{s} - \mathbf{s_g}$ is orthogonal to $\mathbf{g}$.

(c) Show that the length of the projection $\mathbf{s_g}$ of a vector $\mathbf{s}$ onto another vector $\mathbf{g}$ does not exceed the length of $\mathbf{s}$:

$$\|\mathbf{s_g}\| \le \|\mathbf{s}\|. \tag{1}$$

Note that this result generalizes the following 2-D picture:

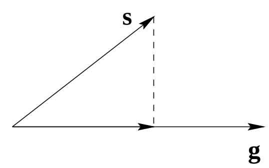

Figure 2: A vector cannot be shorter than its orthogonal projection onto another vector.

(**Hint.** Use the fact that $\mathbf{s} - \mathbf{s_g}$ is orthogonal to $\mathbf{s_g}$. This means that you can apply Pythagoras's theorem proved in (a), to the right triangle whose sides are $\mathbf{s} - \mathbf{s_g}$ and $\mathbf{s_g}$, and whose hypotenuse is $\mathbf{s}$.)

(d) Prove the Cauchy-Schwartz inequality:

$$|\langle \mathbf{g}, \mathbf{s} \rangle| \le \|\mathbf{g}\| \|\mathbf{s}\|$$

(**Hint.** Use inequality (1), proved in (c). In (1), write $\mathbf{s_g}$ in terms of $\mathbf{s}$ and $\mathbf{g}$. Then use $\|a\mathbf{g}\| = |a| \|g\|$, and cancel some terms.)

(e) Prove the triangle inequality: the sum of the lengths of two sides of a triangle cannot be smaller than the length of its third side, i.e.,

$$\|\mathbf{s}\| + \|\mathbf{g}\| \ge \|\mathbf{s} + \mathbf{g}\|$$

Note that this result generalizes the following 2-D picture:

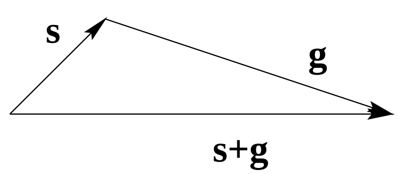

Figure 3: A side of a triangle cannot be longer than the sum of the lengths of the two other sides, i.e., $\|\mathbf{s} + \mathbf{g}\| \le \|\mathbf{s}\| + \|\mathbf{g}\|$.

(**Hint.** Square both sides of the inequality. Then use the fact that $\|\mathbf{s} + \mathbf{g}\|^2 = \langle \mathbf{s} + \mathbf{g}, \mathbf{s} + \mathbf{g} \rangle$, and write this out. Then think how to use the Cauchy-Schwartz inequality proved in (c).

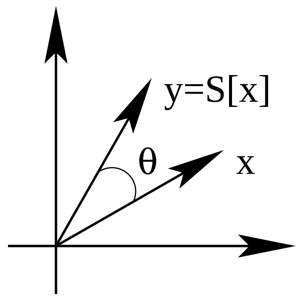

Figure 4: Illustration to Problem 1: in 2-D, the system S rotates the input by some angle θ.

Problem 2. Geometric interpretation of Fourier series.
The following system S takes 2-point real-valued signals as inputs, and produces 2-point signals as outputs. The system is specified by the following input-output relationship:

$$y(n) = \frac{1}{\sqrt{2}} \sum_{k=1}^{2} x(k) \exp\left(j\pi(k-1)(2-n)\right), \text{ for } n = 1, 2,$$

where $\mathbf{y} = S[\mathbf{x}] = \begin{pmatrix} y(1) \\ y(2) \end{pmatrix}$ is the response of the system to the input $\mathbf{x} = \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} \in \mathbb{R}^2$.

(a) Show that S is a rotation of the plane $\mathbb{R}^2$. In other words, show that, for any vector $\mathbf{x} = \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} \in \mathbb{R}^2$, the response of the system $\mathbf{y} = S[\mathbf{x}]$ is the rotation of $\mathbf{x}$ by a certain angle θ (see Fig. 4), where θ is the same for all inputs $\mathbf{x}$. Find θ.

Hints. One method would be to write $\mathbf{x}$ in polar coordinates: $x(1) = R\cos\phi$, $x(2) = R\sin\phi$, and consider the vector $\mathbf{x}_\theta$, the result of rotating $\mathbf{x}$ counterclockwise by angle θ:

$$x_\theta(1) = R\cos(\phi+\theta), \tag{2}$$
$$x_\theta(2) = R\sin(\phi+\theta). \tag{3}$$

Now use trigonometric identities (such as $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$) to massage the two expressions (2) and (3), and write $\mathbf{x}_\theta$ in terms of $\mathbf{x}$. You should get an expression of the form:

$$\mathbf{x}_\theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mathbf{x}, \tag{4}$$

where the coefficients A, B, C, and D depend on θ. Go back to the definition of the system S, and write it in the same form:

$$\mathbf{y} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \mathbf{x}, \tag{5}$$

where A_1, B_1, C_1, and D_1 are numbers. Now find θ by identifying coefficients in Eq. (4) and Eq. (5), and prove that Eq. (5) indeed describes a rotation. It may be helpful to draw a few pictures in the plane before and during your calculations.

(b) Let $\mathbf{x}$ and $\mathbf{v}$ be two real-valued 2-point input signals, such that the distance between them is 7:

$$\sqrt{(x(1)-v(1))^2 + (x(2)-v(2))^2} = 7.$$

Let $\mathbf{y}$ and $\mathbf{w}$ be the responses of S to these signals, respectively: $\mathbf{y} = S[\mathbf{x}]$, and $\mathbf{w} = S[\mathbf{v}]$. Calculate the following quantity:

$$\sqrt{(y(1) - w(1))^2 + (y(2) - w(2))^2}.$$

Suppose that the angle between $\mathbf{x}$ and $\mathbf{v}$ is $5\pi/8$. What is the angle between $\mathbf{y}$ and $\mathbf{w}$? For all parts, **justify your answers**.

Problem 3. Consider system S specified by the following input-output relationship:

$$y(n) = x * h(n),$$

where $h(n)$ is a *fixed* signal. In other words, the response of system S to any input x is the convolution of x with a fixed signal h. Argue that this system is LTI.

Problem 4. Calculate the DT convolution $x * h(n)$ for the following pairs of signals:

(a) $x(n) = e^{-n}u(n)$, $h(n) = \begin{cases} \frac{1}{20}, & n = 0, 1, \ldots, 19 \\ 0, & \text{otherwise.} \end{cases}$

(b) $x(n) = e^{-n}u(n)$, $h(n) = \begin{cases} \frac{1}{20}, & n = -20, -19, \ldots, -1 \\ 0, & \text{otherwise.} \end{cases}$

(c) $x(n) = e^{-n}u(n)$, $h(n) = \begin{cases} 1, & n = -20, -19, \ldots, -1 \\ 2, & n = 0, 1, \ldots, 19 \\ 0, & \text{otherwise.} \end{cases}$

Hint. You should not have to evaluate the convolution sums for Parts (b) and (c). Instead, modify your answer to Part (a) using the result of Problem 3.