NAME:___________________________________

ECE 438
Exam 1 Solutions, 10/06/2004.

- This is a closed-book exam, but you are allowed one standard (8.5-by-11) sheet of notes. No calculators are allowed.
- Total number of points: 105; in addition, 10 bonus points will be awarded for getting 100% on any two of the first six problems. This exam counts for 20% of your final grade.
- You have 75 minutes to complete NINE problems.
- Be sure to **fully and clearly** explain all your answers.
- There will not be any discussion of grades. All re-grade requests must be submitted in writing, as stated in the course information handout.

Score	Grader
1_______	_______
2_______	_______
3_______	_______
4_______	_______
5_______	_______
6_______	_______
bonus_______	
7_______	_______
8_______	_______
9_______	_______

Total score:_______

Formulas which you may find useful:

$$\begin{aligned}
\text{DTFT} \quad X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\
\text{IDTFT} \quad x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \\
\text{DFT} \quad X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\
\text{IDFT} \quad x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}
\end{aligned}$$

Problem 1 (5 points). The impulse response of a discrete-time LTI system is the unit step signal, $u(n)$. Is the system BIBO stable? Fully justify your answer.

Solution. Since the impulse response is not absolutely summable, the system is not BIBO stable.

Problem 2 (5 points). Is the discrete-time signal $x(n) = \sin\left(\frac{\pi}{4}n\right)$ periodic? If so, find its fundamental period.

Solution. The plot of this signal is shown in Problem 8. Since $\sin\left(\frac{\pi}{4}(n+8k)\right) = \sin\left(\frac{\pi}{4}n + 2\pi k\right) = \sin\left(\frac{\pi}{4}n\right)$ for any integer k, this signal is periodic. It is evident from the plot that the fundamental period is 8.

Problem 3 (5 points). The frequency response of an LTI system is:

$$H(e^{j\omega}) = \frac{1}{1 + 0.5e^{-j\omega}}.$$

Find the response of this system to the following input signal:

$$x(n) = e^{j\pi n} \quad \text{for all integer } n.$$

Solution. Since the input signal is an everlasting exponential of frequency $\omega = \pi$, the response will be equal to the input signal multiplied by the frequency response at $\omega = \pi$:

$$H(e^{j\pi})x(n) = \frac{e^{j\pi n}}{1 + 0.5e^{-j\pi}} = 2e^{j\pi n}.$$

Problem 4 (5 points). Let

$$X(e^{j\omega}) = \begin{cases} e^{-2j\omega}, & |\omega| < 1, \\ 0, & 1 \le |\omega| \le \pi. \end{cases}$$

Evaluate $x(n)$, the inverse DTFT of $X(e^{j\omega})$. Simplify your answer as much as possible.

Solution.

$$\frac{1}{2\pi}\int_{-1}^{1} e^{-2j\omega}e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-1}^{1} e^{j(n-2)\omega}\,d\omega = \frac{1}{2\pi}\cdot\frac{1}{j(n-2)}\left(e^{j(n-2)} - e^{-j(n-2)}\right) = \frac{1}{\pi}\cdot\frac{\sin(n-2)}{n-2}.$$

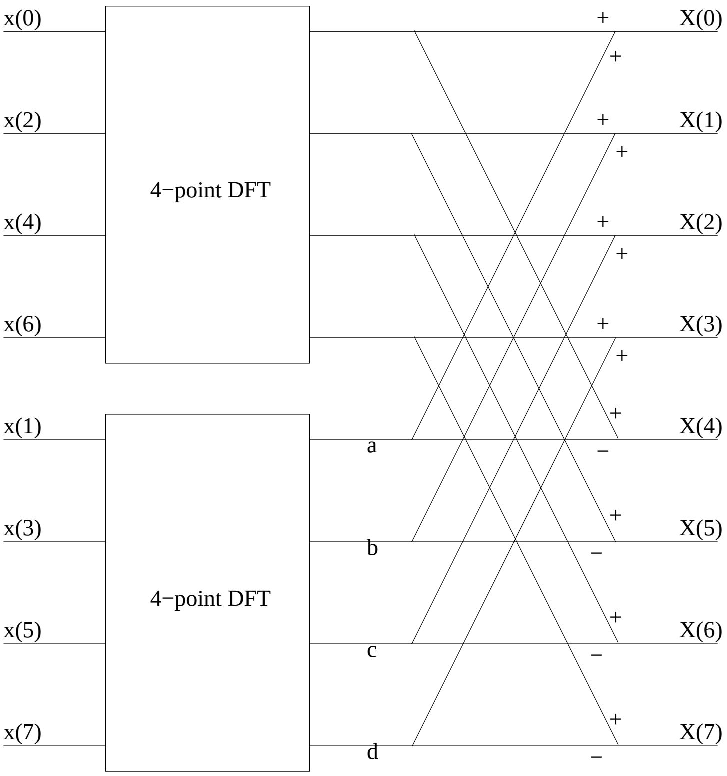

Figure 1: Diagram for Problem 5.

Problem 5 (5 points). Specify the numbers a, b, c, d, such that the diagram in Fig. 1 implements the 8-point DFT of $x(n)$.

Solution. The answer is W_8^k for $k = 0, 1, 2, 3$, i.e.:

$$\begin{aligned} a &= \left(e^{-j2\pi/8}\right)^0 = 1, \\ b &= \left(e^{-j2\pi/8}\right)^1 = e^{-j\pi/4}, \\ c &= \left(e^{-j2\pi/8}\right)^2 = e^{-j\pi/2} = -j, \\ d &= \left(e^{-j2\pi/8}\right)^3 = e^{-j3\pi/4}. \end{aligned}$$

Problem 6 (5 points). It is known that $x(n) * u(n) = 3^{n+1}$ for all integer n, where $*$ stands for discrete-time convolution, and $u(n)$ is the discrete-time unit step. Find $x(n)$ for all integer n.

Solution. Note that

$$x(n) = x(n) * \delta(n) = x(n) * [u(n) - u(n-1)] = x(n) * u(n) - x(n) * u(n-1).$$

Since the convolution of $x(n)$ with $u(n)$ is 3^{n+1}, the convolution $x(n)$ with $u(n-1)$ is 3^n:

$$x(n) = 3^{n+1} - 3^n = 2 \cdot 3^n.$$

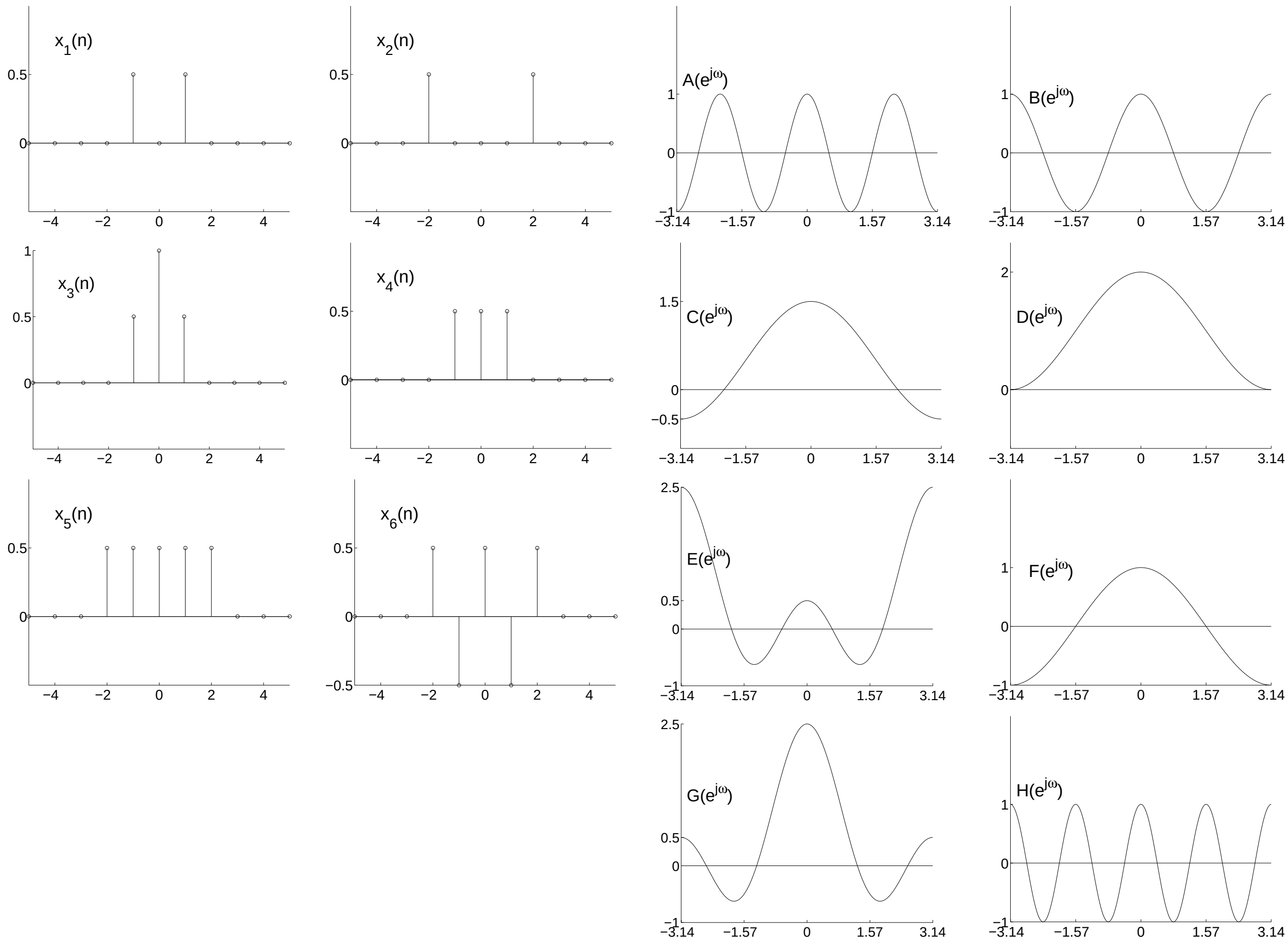

Figure 2: Signals (two left columns) and DTFT's (two right columns) for Problem 7.

Problem 7 (30 points). Discrete-time signals $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$, $x_5(n)$, and $x_6(n)$ are shown in the first two columns of Fig. 2. They are all plotted for $n = -5, -4, \ldots, 5$. All these signals are zero outside of this range. DTFT's $A(e^{j\omega})$, $B(e^{j\omega})$, $C(e^{j\omega})$, $D(e^{j\omega})$, $E(e^{j\omega})$, $F(e^{j\omega})$, $G(e^{j\omega})$, and $H(e^{j\omega})$ are shown in the last two columns. Note that the DTFT's are all plotted for $|\omega| \le \pi$ and are periodic with period 2π. Six of the eight DTFT's correspond to the six given signals. For each of the six signals, locate its DTFT in the last two columns and **fully explain your choice**.

Solution. The DTFT of $0.5(\delta(n-n_0)+\delta(n+n_0))$ is $\cos(n_0\omega)$, therefore $X_1(e^{j\omega}) = F(e^{j\omega})$ and $X_2(e^{j\omega}) = B(e^{j\omega})$. Since $x_3(n) = \delta(n) + x_1(n)$, $X_3(e^{j\omega}) = 1 + X_1(e^{j\omega}) = D(e^{j\omega})$. Similarly, $x_4(n) = 0.5\delta(n) + x_1(n)$, and therefore $X_4(e^{j\omega}) = 0.5 + X_1(e^{j\omega}) = C(e^{j\omega})$. Since $x_5(n) = x_2(n) + x_4(n)$, we have: $X_5(e^{j\omega}) = X_2(e^{j\omega}) + X_4(e^{j\omega}) = B(e^{j\omega}) + C(e^{j\omega})$. To determine which one this is, look at $\omega = 0$: $X_5(e^{j\cdot 0}) = B(e^{j\cdot 0}) + C(e^{j\cdot 0}) = 1 + 1.5 = 2.5$, which means that the only possibility for $X_5(e^{j\omega})$ is $G(e^{j\omega})$. Finally, $x_6(n) = x_5(n) - 2x_1(n)$, therefore $X_6(e^{j\omega}) = X_5(e^{j\omega}) - 2X_1(e^{j\omega}) = G(e^{j\omega}) - 2F(e^{j\omega})$. To determine which one this is, we can again look at $\omega = 0$: $X_6(e^{j\cdot 0}) = G(e^{j\cdot 0}) - 2F(e^{j\cdot 0}) = 2.5 - 2 = 0.5$, and the only possibility is $E(e^{j\omega})$.

Answer: (x_1, F), (x_2, B), (x_3, D), (x_4, C), (x_5, G), (x_6, E). A and H do not correspond to any of the six given signals.

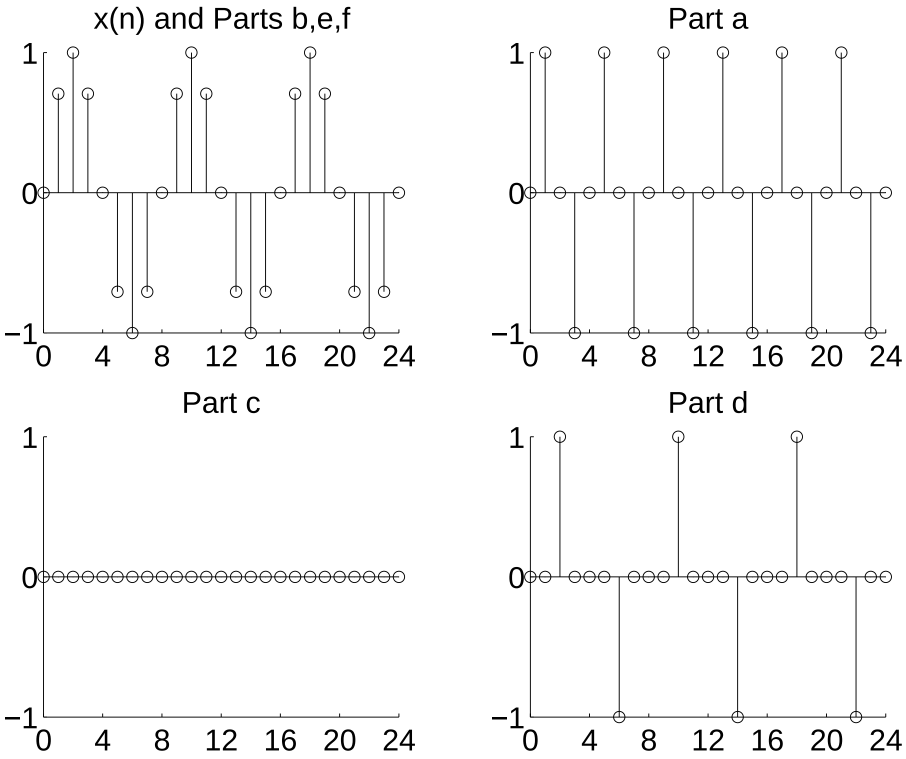

Figure 3: Signal $x(n)$ for Problem 8.

Problem 8 (30 points). The signal $x(n) = \sin\left(\frac{\pi}{4}n\right)$ is shown in Fig. 3. Sketch the result of each of the following transformations applied to this signal. Fully label the axes and explain your answers.

a. Downsampling $x(n)$ by a factor of 2.

b. Low-pass filtering $x(n)$ with an ideal lowpass filter whose frequency response is

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{3}, \\ 0, & \text{otherwise.} \end{cases}$$

c. First downsampling $x(n)$ by a factor of 2, and then lowpass filtering the result with the filter defined in Part b.

d. First downsampling $x(n)$ by a factor of 2, and then upsampling the result by a factor of 2.

e. First upsampling $x(n)$ by a factor of 2, and then downsampling the result by a factor of 2.

f. Putting $x(n)$ through the system described by the following input-output relationship:

$$y(n) = x(n-8).$$

(Recall that downsampling = discarding some samples, and upsampling = inserting zeros.)

Solution.

a. Downsampling by a factor of 2 produces $x_d(n) = x(2n) = \sin(\pi n/2)$, shown in the top right plot of Fig. 3.

b. Since the frequency of the original sinusoid, $\pi/4$, is less than $\pi/3$, the cutoff frequency of the lowpass filter, the sinusoid will pass through the filter unchanged. The result is shown in the upper left plot of Fig. 3.

c. Here we need to determine what happens if $x_d(n) = \sin(\pi n/2)$ is put through the same filter. Since the frequency of this sinusoid, $\pi/2$, is above the cutoff frequency of the filter, the sinusoid will be completely filtered out, and the response will be the zero signal, shown in the lower left corner of Fig. 3.

d. Now $x_d(n) = \sin(\pi n/2)$ is upsampled by a factor of 2. In this case, an extra zero will be inserted between every pair of consecutive samples, as shown in the lower right corner of Fig. 3.

e. If the order of the upsampling and downsampling is reversed, the result will be different: namely, it will be the original signal. Indeed, if the original signal is upsampled, a zero will be inserted after each of its samples. Then, when the resulting signal is downsampled, the zeros will be discarded.

f. Since the signal is periodic with period 8, shifting it by 8 samples will leave it unchanged.

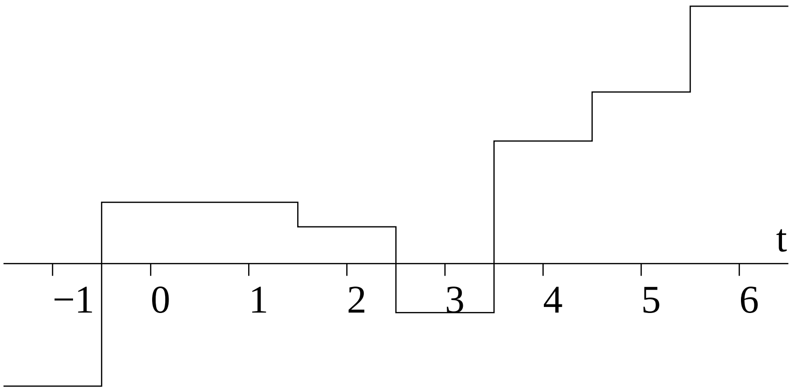

Figure 4: A staircase signal.

Problem 9 (15 points). Consider the set G of all continuous-time "staircase" signals which are constant over intervals $\ldots, -0.5 \leq t < 0.5$, $0.5 \leq t < 1.5$, $1.5 \leq t < 2.5, \ldots$. An example of such a signal is shown in Fig. 4.

a. Suppose that a discrete-time signal $x(n)$ consists of samples of a continuous-time signal $x_c(t)$, i.e., $x(n) = x_c(nT_s)$ where T_s is the sampling period. Suppose further that it is known that $x_c(t)$ is in the set G. What is the largest T_s for which is it possible to perfectly reconstruct $x_c(t)$ from its samples? Describe this reconstruction procedure. (**Hint:** sketch the samples of the signal in Fig. 4 and use common sense. Do not consider the frequency domain.)

b. Suppose now we sample some signal $v(t)$ which is not necessarily in the set G, with sampling period $T_s = 1$. We use the reconstruction procedure you developed in Part a, to reconstruct a continuous-time staircase signal $y(t)$. Note that if $v(t)$ is not in G, then $y(t)$ will not be the same signal as $v(t)$. Describe a prefiltering procedure to be performed before sampling, in order to insure that the reconstructed signal $y(t)$ is as close as possible to $v(t)$, i.e., that the energy of the error, $\int_{-\infty}^{\infty} (v(t) - y(t))^2 \, dt$, is minimized.

Solution.

a. Since the continuous-time signal is constant over $n - 0.5 \leq t < n + 0.5$, we need at least one sample from each such interval in order to be able to perfectly reconstruct, i.e., the largest T_s is 1. In this case, the original signal is recovered through: $x_c(t) = x(n)$ for $n - 0.5 \leq t < n + 0.5$.

b. Let a_n be the value of the reconstructed signal $y(t)$ on the interval $n - 0.5 \leq t < n + 0.5$. The energy of the error over this interval is then $E(a_n) = \int_{n-0.5}^{n+0.5} (v(t) - a_n)^2 \, dt$. To minimize, we take the derivative and set it to zero:

$$\begin{aligned} \frac{dE}{da_n} &= \int_{n-0.5}^{n+0.5} 2(a_n - v(t)) \, dt = 0 \\ a_n &= \int_{n-0.5}^{n+0.5} v(t) \, dt \end{aligned}$$

Since the second derivative is 2, this is a minimum. Thus, in order to minimize the energy of the error, the sample at n must be equal to the average value of $v(t)$ over the interval $n - 0.5 \leq t < n + 0.5$. This can be accomplished by prefiltering $v(t)$ with a filter whose impulse response is:

$$h(t) = \begin{cases} 1, & |t| < 0.5, \\ 0, & \text{otherwise.} \end{cases}$$