NAME:___________________________________

ECE 438
Exam 2 Solutions, 11/16/2004.

- This is a closed-book exam, but you are allowed one standard (8.5-by-11) sheet of notes. No calculators are allowed.
- Total number of points: 110; in addition, 10 bonus points will be awarded for getting at least 50% on the first seven problems. This exam counts for 20% of your final grade.
- You have 75 minutes to complete NINE problems.
- Be sure to **fully and clearly** explain all your answers.
- There will not be any discussion of grades. All re-grade requests must be submitted in writing, as stated in the course information handout.

Score	Grader
1_______	_______
2_______	_______
3_______	_______
4_______	_______
5_______	_______
6_______	_______
7_______	_______
bonus_______	
8_______	_______
9_______	_______

Total score:_______

Problem 1 (5 points). Find the z-transform of the following discrete-time signal: $x(n) = \delta(n) + \delta(n-1)$.

Solution. Since the z-transform of δ is 1, and since shifting by 1 in the time domain corresponds to multiplication by z^{-1} in the z-domain, we have: $X(z) = 1 + z^{-1}$.

Problem 2 (5 points). Find the transfer function of the LTI system specified by the following input-output relationship: $y(n) = x(n) - x(n-1)$, for all n.

Solution. $H(z) = Y(z)/X(z) = 1 - z^{-1}$.

Problem 3 (5 points). The z-transform of a signal $x(n)$ is $X(z) = \frac{3}{1-0.2z^{-1}}$, with region of convergence $|z| > 0.2$. Find $x(n)$ for all n.

Solution. As shown in class, the inverse z-transform is $3 \cdot 0.2^n u(n)$.

Problem 4 (15 points). The transfer function of a causal LTI system is:

$$H(z) = \frac{1}{1 + 0.5z^{-1}}.$$

(a) Is this system BIBO stable? Fully justify your answer.

(b) Find the response of this system to the following input signal:

$$x(n) = 2^n \quad \text{for all integer } n.$$

(c) Suppose the input to this system is a zero-mean, wide-sense stationary random process. Find the mean of the output. Fully justify your answer.

Solution. The transfer function has one pole, at $z = -0.5$. Since the system is causal, the region of convergence for this transfer function must be outside of the pole, i.e., $|z| > 0.5$, which means that the unit circle is within the region of convergence, which, in turn, means that the system is BIBO stable.

Since the signal $x(n) = 2^n$ is an everlasting exponential, and since $z = 2$ is inside the region of convergence, the response is $2^n H(2) = 0.8 \cdot 2^n$.

If the input is a WSS process with mean μ, then the mean of the output is $\mu \sum_n h(n) = \mu H(1)$, as long as the system is BIBO stable. In our case, $\mu = 0$, and so the mean of the output is also 0.

Problem 5 (20 points). Let $X_1, X_2, \dots, X_{10}$ be ten real-valued, continuous, independent random variables, each having the following uniform probability density function:

$$f(x) = \begin{cases} 1/2, & \text{if } 1 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} Y &= \frac{1}{10}\sum_{n=1}^{10} X_n. \\ W &= \frac{1}{10}\sum_{n=1}^{10} X_n^2. \\ V &= X_1 \cdot X_2 \cdot \ldots \cdot X_{10}. \end{aligned}$$

(a) Find $E[Y]$.

(b) Find $\text{var}(Y)$.

(c) Find $E[W]$.

(d) Find $E[V]$.

Solution.

(a) Since Y is the sample mean of the X_n's, and since the expectation of each X_n is 2, $E[Y] = E[X_n] = 2$.

(b) Using Eq. (2) from Problem 8 below, we have: $\text{var}(X) = 1/3$. Therefore, $\text{var}(Y) = 1/10\text{var}(X_n) = 1/30$.

(c) $E[X_n^2] = \text{var}(X_n) + (E[X_n])^2 = 1/3 + 4 = 13/3$, therefore, $E[W] = E[X_n^2] = 13/3$.

(d) Since X_n's are independent, the expectation of their product is equal to the product of the individual expectations, i.e., $E[V] = 2^{10} = 1024$.

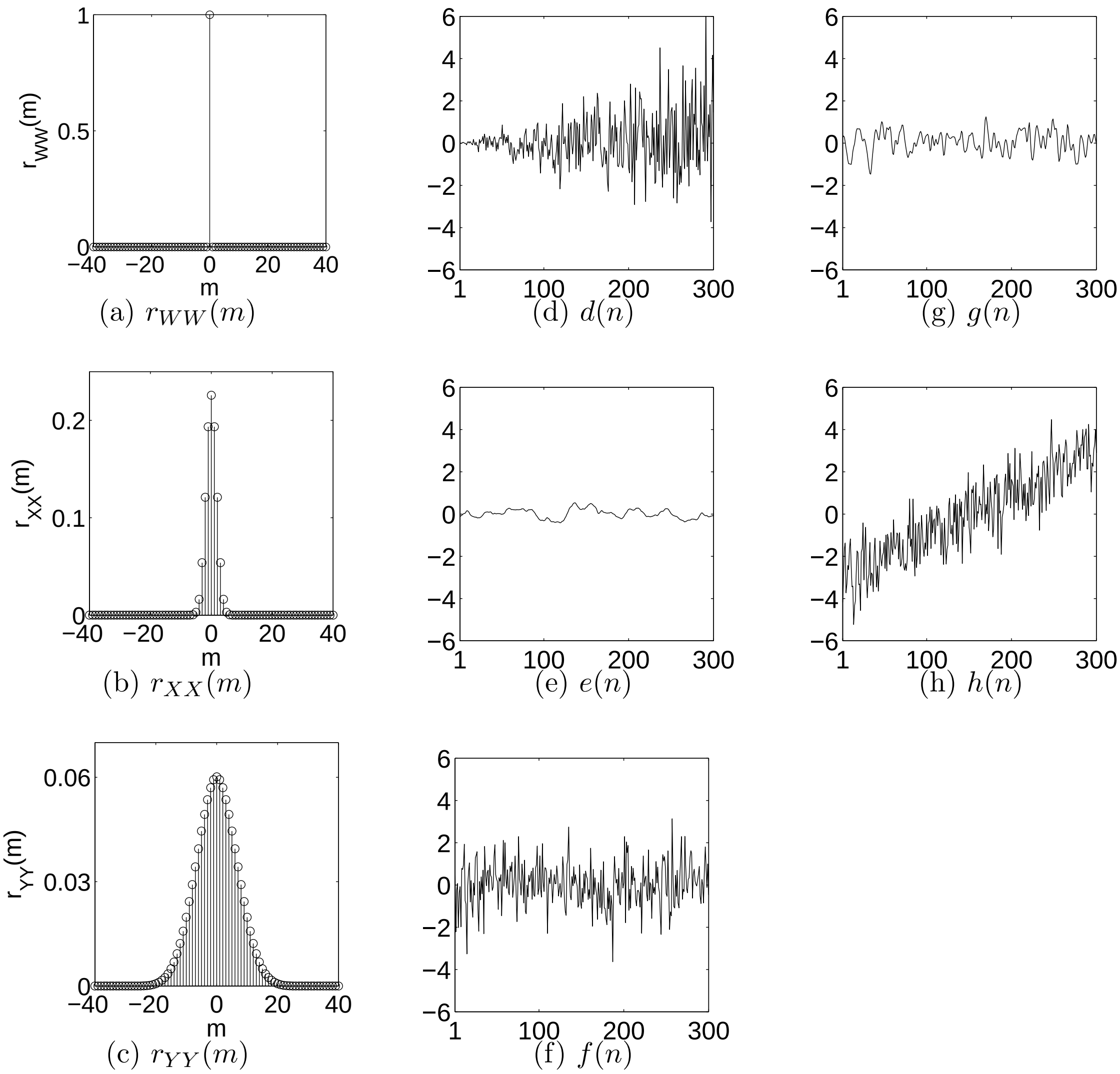

Figure 1: Illustration to Problem 6.

Problem 6 (15 points). The random processes $W(n)$, $X(n)$, and $Y(n)$ are zero-mean, real-valued, wide-sense stationary processes whose autocorrelation functions are depicted in the left column of Fig. 1. Among the five signals $d(n)$, $e(n)$, $f(n)$, $g(n)$, and $h(n)$ in the right two columns, exactly one is a realization of $W(n)$, exactly one is a realization of $X(n)$, and exactly one is a realization of $Y(n)$. Match the three random processes with their realizations and fully explain your answers. (Note that, even though $d(n)$, $e(n)$, $f(n)$, $g(n)$, and $h(n)$ are all discrete-time signals, they are displayed here using Matlab's `plot` command, not `stem`.)

Solution. First, note that $d(n)$ and $h(n)$ are not stationary: $d(n)$ has a growing variance whereas $h(n)$ has a growing mean. Of the remaining three processes, the smoothest, $e(n)$, must correspond to $Y(n)$ since $Y(n)$ has the most long-range correlation among its samples, the second smoothest, $g(n)$, is a realization of $X(n)$, and the most jittery one, $f(n)$, is a realization of the white noise process $W(n)$.

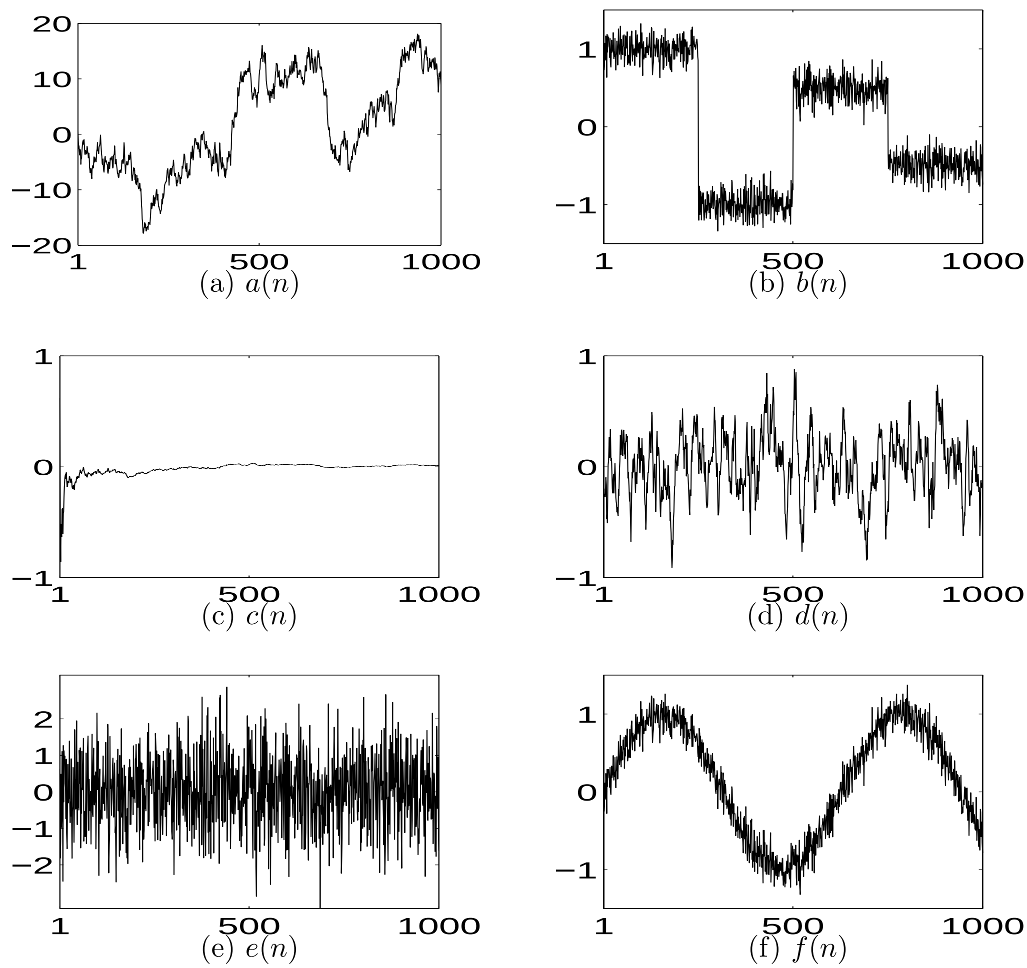

Figure 2: Illustration to Problem 7.

Problem 7 (20 points). $W(n)$ is a zero-mean, real-valued, white noise process, defined for all integer n. Random processes $S(n)$, $V(n)$, and $X(n)$ are defined as follows:

$$\begin{aligned} S(n) &= \frac{1}{10}(W(n) + W(n-1) + \ldots + W(n-9)), \\ V(n) &= W(0) + W(1) + \ldots + W(n), \text{ for } n \geq 0 \\ X(n) &= \frac{V(n)}{n+1}, \text{ for } n \geq 0. \end{aligned}$$

One realization of each of these four processes is depicted in Fig. 2 (in addition, two signals which are not realizations of any of these four processes, are depicted). Match each process with its realization.

Solution. Since $E[X(n)] = 0$ and $var(X(n)) \to 0$ as $n \to \infty$, the realization of $X(n)$ must settle to zero, i.e., it is $c(n)$. Both $W(n)$ and $S(n)$ are WSS, zero-mean processes. The only stationary processes in the figure are $d(n)$ and $e(n)$. Since $var(S(n)) = 0.1var(W(n))$, the standard deviation of $W(n)$ is about three times as large as the standard deviation of $S(n)$, which means that $S(n)$ must correspond to $d(n)$ and $W(n)$ must correspond to $e(n)$. (Another way to establish this is to notice that $W(n)$ is white while each sample of $S(n)$ is correlated with a number of other nearby samples, and therefore a realization of $S(n)$ must be smoother than a realization of $W(n)$.) Finally, the only signal with growing variance is $a(n)$, which must therefore be a realization of $V(n)$.

Problem 8 (15 points). Let A be a random variable with the following pdf:

$$f_A(a) = \begin{cases} 1/8, & \text{if } 0 \le a < 2, \\ 3/8, & \text{if } 2 \le a \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Suppose this random variable is quantized using the following three quantization intervals,

$$[x_0, x_1) = [0, 1), \qquad [x_1, x_2) = [1, 2), \qquad [x_2, x_3] = [2, 4],$$

with the corresponding quantization levels $q_0 = 1/2$, $q_1 = 3/2$, and $q_2 = 3$. Let $\hat{A}$ be the result of quantizing A. Find the MS error, $E[(\hat{A} - A)^2]$.

(b) Now suppose we use the set of following three quantization intervals,

$$[x'_0, x'_1) = [0, 2), \qquad [x'_1, x'_2) = [2, 3), \qquad [x'_2, x'_3] = [3, 4],$$

with quantization levels $q'_0 = 1$, $q'_1 = 5/2$, and $q'_2 = 7/2$. Let $\hat{A}'$ be the result of quantizing A with this quantizer. Is the resulting MS error, $E[(\hat{A}' - A)^2]$, less than, greater than, or equal to the MS error in Part (a)? Fully substantiate your answer. (**Hint.** One way to do this is to actually find $E[(\hat{A}' - A)^2]$; however, there is an easier way.)

(c) Suppose we use the following four-level uniform quantizer:

$$[x''_0, x''_1) = [0, 1), \quad \text{with } q''_0 = 1/2, \qquad [x''_1, x''_2) = [1, 2), \quad \text{with } q''_1 = 3/2,$$
$$[x''_2, x''_3) = [2, 3), \quad \text{with } q''_2 = 5/2, \qquad [x''_3, x''_4] = [3, 4], \quad \text{with } q''_3 = 7/2.$$

Let $\hat{A}''$ be the result of quantizing A with this quantizer.

(i) Is the resulting MS error, $E[(\hat{A}'' - A)^2]$, less than, greater than, or equal to $E[(\hat{A} - A)^2]$ from Part (a)?

(ii) Is the resulting MS error, $E[(\hat{A}'' - A)^2]$, less than, greater than, or equal to $E[(\hat{A}' - A)^2]$ from Part (b)?

Solution. First note the following facts:

$$\int_{-w}^{w} x^2 dx = \left.\frac{x^3}{3}\right|_{-w}^{w} = \frac{2w^3}{3}; \tag{1}$$

$$\int_u^v \left(x - \frac{u+v}{2}\right)^2 dx = \int_{-\frac{v-u}{2}}^{\frac{v-u}{2}} y^2 dy = \frac{2((v-u)/2)^3}{3}, \tag{2}$$

where in Eq. (2) we made a change of variable $y = x - (u+v)/2$ and then used Eq. (1).

(a) The MS error is:

$$\begin{aligned} E[(\hat{A} - A)^2] &= \int_{-\infty}^{\infty} (\hat{a} - a)^2 f_A(a) da \\ &= \int_0^1 \left(\frac{1}{2} - a\right)^2 \frac{1}{8} da + \int_1^2 \left(\frac{3}{2} - a\right)^2 \frac{1}{8} da + \int_2^4 (3 - a)^2 \frac{3}{8} da \\ &\overset{\text{Eq. (2)}}{=} \frac{2(1/2)^3}{3} \cdot \frac{1}{8} + \frac{2(1/2)^3}{3} \cdot \frac{1}{8} + \frac{2 \cdot 1^3}{3} \cdot \frac{3}{8} = \frac{13}{48}. \end{aligned}$$

(b) Note that $P(0 \leq A < 2) = 1/4$ whereas $P(2 \leq A \leq 4) = 3/4$. In Part (a), we used two quantization levels to quantize the lower-probability interval $[0, 2)$, and only one quantization level for the higher-probability interval $[2, 4]$. A smarter strategy is to use more quantization levels for more probable intervals and fewer quantization levels for less probable intervals. Thus, using two quantization levels for $[2, 4]$ and one quantization level for $[0, 2)$ should lead to a smaller MS error. To verify this, note that

$$\begin{aligned} E[(\hat{A}' - A)^2] &= \int_{-\infty}^{\infty} (\hat{a}' - a)^2 f_A(a) da \\ &= \int_0^2 (1-a)^2 \frac{1}{8} da + \int_2^3 \left(\frac{5}{2} - a\right)^2 \frac{3}{8} da + \int_3^4 \left(\frac{7}{2} - a\right)^2 \frac{3}{8} da \\ &\overset{\text{Eq. (2)}}{=} \frac{2 \cdot 1^3}{3} \cdot \frac{1}{8} + \frac{2(1/2)^3}{3} \cdot \frac{3}{8} + \frac{2(1/2)^3}{3} \cdot \frac{3}{8} = \frac{7}{48} < \frac{13}{48}. \end{aligned}$$

(c) This quantizer quantizes the interval $[0, 2)$ in the same way as the quantizer of Part (a), and it quantizes the interval $[2, 4]$ using two quantization levels instead of one. Therefore, it achieves a smaller MS error. Similarly, it quantizes the interval $[2, 4]$ in the same way as the quantizer of Part (b), and uses two levels instead of one for $[0, 2)$. Therefore, it will also achieve a smaller error than the quantizer in Part (b). Its MS error is calculated as follows:

$$\begin{aligned} E[(\hat{A}'' - A)^2] &= \int_{-\infty}^{\infty} (\hat{a}'' - a)^2 f_A(a) da \\ &= \int_0^1 \left(\frac{1}{2} - a\right)^2 \frac{1}{8} da + \int_1^2 \left(\frac{3}{2} - a\right)^2 \frac{1}{8} da \\ &+ \int_2^3 \left(\frac{5}{2} - a\right)^2 \frac{3}{8} da + \int_3^4 \left(\frac{7}{2} - a\right)^2 \frac{3}{8} da \\ &\overset{\text{Eq. (2)}}{=} \frac{2(1/2)^3}{3} \left(\frac{1}{8} + \frac{1}{8} + \frac{3}{8} + \frac{3}{8}\right) = \frac{1}{12} = \frac{4}{48} < \frac{7}{48}. \end{aligned}$$

Problem 9 (10 points). $X_1, X_2, \ldots, X_N$ are independent, identically distributed random variables whose marginal pdf depends on a parameter k in the following way:

$$\begin{aligned} f_{X|k}(x|k=0) &= \begin{cases} 1, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases} \\ f_{X|k}(x|k=1) &= \begin{cases} 2x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(a) Find the maximum likelihood estimate $\hat{k}$ of k based on observing $X_1 = x_1, X_2 = x_2, \ldots, X_N = x_N$, i.e., specify a rule for choosing $\hat{k} = 0$ or $\hat{k} = 1$ so as to maximize $f_{X_1,\ldots,X_N|k}(x_1, \ldots, x_N|k)$.

(b) If $N = 1$, find $P(error|k=0)$ and $P(error|k=1)$ for the estimate in Part (a).

Solution. Because of independence, joint density is equal to the product of the marginals, and so

$$\begin{aligned} f_{X_1,\ldots,X_N|k}(x_1, \ldots, x_N|k=0) &= \prod_{n=1}^{N} f_{X|k}(x_n|k=0) = 1; \\ f_{X_1,\ldots,X_N|k}(x_1, \ldots, x_N|k=1) &= \prod_{n=1}^{N} f_{X|k}(x_n|k=1) = 2^N \prod_{n=1}^{N} x_n. \end{aligned}$$

The maximum likelihood rule is therefore to say that $\hat{k} = 0$ if $\prod_{n=1}^{N} x_n < 2^{-N}$ and to say that $\hat{k} = 1$ otherwise.

If we do so, then, for $N = 1$,

$$\begin{aligned} P(error|k=0) &= P(\hat{k}=1|k=0) = P(x_1 \geq 1/2 \,|\, k=0) = \int_{0.5}^{1} dx = 1/2. \\ P(error|k=1) &= P(\hat{k}=0|k=1) = P(x_1 < 1/2 \,|\, k=1) = \int_{0}^{0.5} 2x dx = 1/4. \end{aligned}$$