NAME:___________________________________

ECE 438
Final Exam Solutions, 12/14/2004.

- This is a closed-book exam, but you are allowed two standard (8.5-by-11) sheets of notes. No calculators are allowed.
- Total number of points: 135. This exam counts for 25% of your final grade.
- You have 120 minutes to complete seven problems.
- Be sure to **fully and clearly** explain all your answers.
- The graded exams will be available to pick up starting Monday December 20 in MSEE 330 during regular business hours.

Score	Grader
1_______	_______
2_______	_______
3_______	_______
4_______	_______
5_______	_______
6_______	_______
7_______	_______

Total score:_______

Problem 1 (35 points). A discrete-time system is defined by the following input-output relation:

$$y(n) = x(2n), \text{ for all integer } n.$$

(**Hint.** This system performs downsampling.)

a. (5 points) Find the impulse response of this system—i.e., find the output of the system if the input signal is $\delta(n)$.

   Solution. Since $\delta(2n) = \delta(n)$, the impulse response of this system is $\delta(n)$.

b. (5 points) Find the response of this system to the signal $\delta(n-1)$.

   Solution. Note that the output consists of the even-numbered input samples. Since all the even-numbered samples of $\delta(n-1)$ are zeros, the output will be zero for all n.

c. (5 points) Is this system time-invariant?

   Solution. From Part (a), the response to $\delta(n)$ is $h(n) = \delta(n)$; from Part (b), the response to $\delta(n-1)$ is zero, which is not equal to $h(n-1)$. Therefore, the system is not time-invariant.

d. (5 points) Is this system causal?

   Solution. Since $y(1)$ depends on $x(2)$, the system is not causal.

e. (5 points) Is this system BIBO stable?

   Solution. If $|x(n)| < M$ for all n, then certainly also $|y(n)| = |x(2n)| < M$ for all n, and therefore the system is BIBO stable.

f. (5 points) Suppose that the DTFT of the input signal for $-\pi \le \omega \le \pi$ is:

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{3}, \\ 0, & \frac{\pi}{3} < |\omega| \le \pi. \end{cases}$$

   Find $Y(e^{j\omega})$, the DTFT of the output signal, for $-\pi \le \omega \le \pi$.

   Solution. Since the spectrum is zero for $|\omega| > \pi/2$, downsampling by 2 results in no aliasing. The spectrum in $-\pi \le \omega \le \pi$ will be stretched by 2 and multiplied by 1/2:

$$Y(e^{j\omega}) = \begin{cases} 1/2, & |\omega| \le \frac{2\pi}{3}, \\ 0, & \frac{2\pi}{3} < |\omega| \le \pi. \end{cases}$$

g. (5 points) Suppose the input to this system is a zero-mean, wide-sense stationary random process with autocorrelation function $r_{XX}(m) = 0.5\delta(m-1) + \delta(m) + 0.5\delta(m+1)$. Will the output be wide-sense stationary? Find the autocorrelation function r_{YY} of the output.

   Solution. Since $E(Y(n)) = E(X(2n)) = 0$ and $E(Y(n)Y(n+m)) = E(X(2n)X(2n+2m)) = r_{XX}(2m)$, Y is wide-sense stationary. Also, $r_{YY}(m) = r_{XX}(2m) = \delta(m)$.

Problem 2 (10 points). Let $Y(n) = X(n) + X(n-1) + X(n-2)$ where $X(n)$ is a zero-mean, wide-sense stationary random process with autocorrelation function $r_{XX}(m) = \delta(m)$.

a. (5 points) Find the autocorrelation function r_{YY} of $Y(n)$.

   Solution. The answer is $h * h_-(m)$ where $h(n) = \delta(n)+\delta(n-1)+\delta(n-2)$ and $h_-(n) = h(-n)$. Performing the convolution yields: $r_{YY}(m) = \delta(m+2)+2\delta(m+1)+3\delta(m)+2\delta(m-1)+\delta(m-2)$.

b. (5 points) Find the coefficient a for the first-order linear predictor

$$\hat{Y}(n) = aY(n-1)$$

   which minimizes the MS error, $E[(\hat{Y}(n) - Y(n))^2]$.

   Solution.

$$\begin{aligned}
\hat{Y}(n) &= \text{projection of } Y(n) \text{ on } Y(n-1) \\
&= \frac{\langle Y(n), Y(n-1)\rangle}{\langle Y(n-1), Y(n-1)\rangle} Y(n-1) \\
&= \frac{r_{YY}(1)}{r_{YY}(0)} Y(n-1) \\
&= \frac{2}{3} Y(n-1).
\end{aligned}$$

   Therefore, $a = 2/3$.

Problem 3 (15 points). Compute the short-time Fourier transform

$$X(m, e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)w(n-m)e^{-j\omega n}$$

for the signal

$$x(n) = 2^n u(n)$$

and a 3-point rectangular window:

$$w(n) = \delta(n) + \delta(n-1) + \delta(n-2).$$

Here, as usual, $u(n)$ is the discrete-time unit step function, and $\delta(n)$ is the discrete-time unit impulse.

Solution. Consider four cases:

(1) $m \le -3$: in this case, the support of the window function $w(n-m)$ and the support of $x(n)$ do not overlap. Therefore $X(m, e^{j\omega}) = 0$.

(2) $m = -2$: in this case, the support of the window function overlaps with the support of x at exactly one point, $n = 0$:

$$X(-2, e^{j\omega}) = \sum_{n=0}^{0} 2^n e^{-j\omega n} = 1.$$

(3) $m = -1$: in this case, the support of the window function overlaps with the support of x at exactly two points, $n = 0, 1$:

$$X(-1, e^{j\omega}) = \sum_{n=0}^{1} 2^n e^{-j\omega n} = 1 + 2e^{-j\omega}.$$

(4) $m \ge 0$: in this case, the entire support of the window function is inside the support of x.

$$\begin{aligned} X(m, e^{j\omega}) &= \sum_{n=m}^{m+2} 2^n e^{-j\omega n} \\ &= 2^m e^{-j\omega m} + 2^{m+1} e^{-j\omega(m+1)} + 2^{m+2} e^{-j\omega(m+2)} \\ &= (2e^{-j\omega})^m (1 + 2e^{-j\omega} + 4e^{-2j\omega}) \\ &= (2e^{-j\omega})^m \frac{1 - \left(2e^{-j\omega}\right)^3}{1 - 2e^{-j\omega}}. \end{aligned}$$

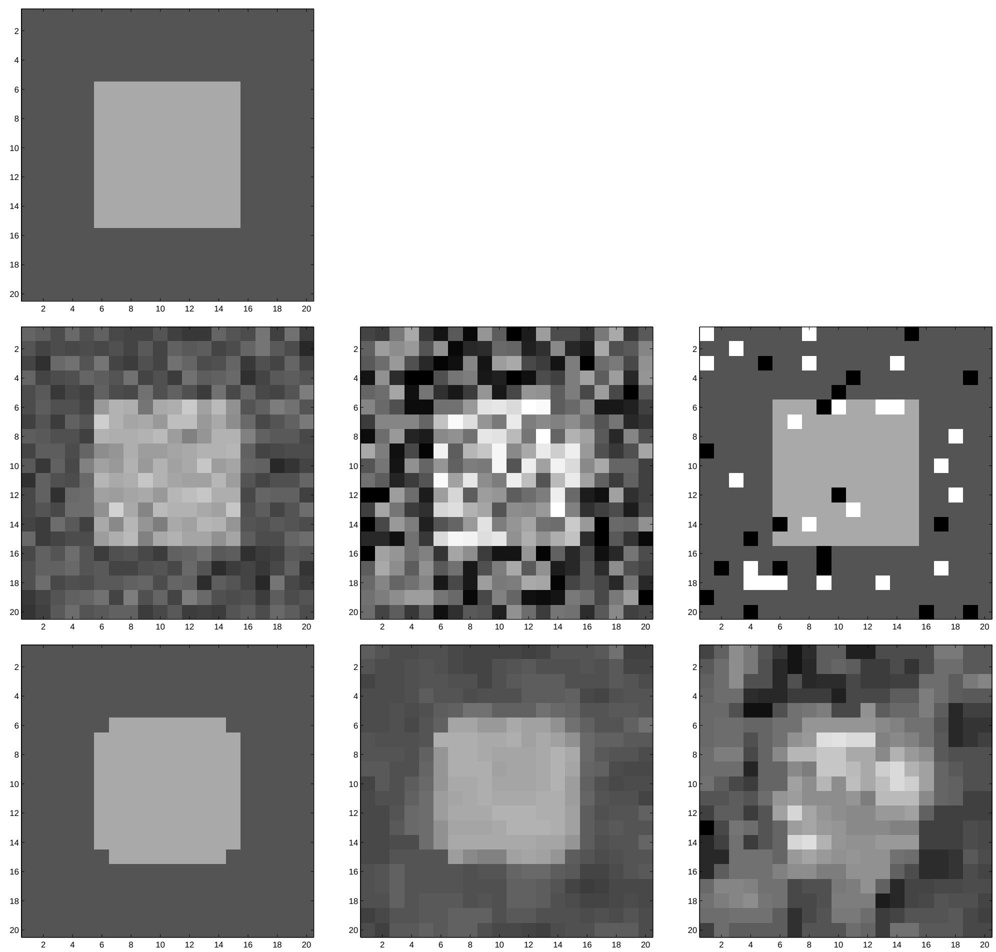


Figure 1: Images for Problem 4.

Problem 4 (30 points). In Fig. 1, all the images are 20×20 discrete-space, continuous-valued, grayscale images. Each pixel can have values from 0 (black) to 3 (white). The image in the top row depicts a light square on a dark background. Within the square, the image has a constant value of 2; within the background, it has a constant value of 1. (Note: the images look somewhat grainy and non-constant on your printout because of imperfections of printing and copying devices.) Each of the six images in the second and third rows was obtained by modifying the square image in the top row. The objective of this problem is to determine what operations resulted in the six images.

a. (15 points) One of the images in the second row was obtained from the square image in the top row by adding white zero-mean noise with marginal density which is uniform between -1 and 1. One of the images in the second row was obtained from the square image in the top row by adding white zero-mean Gaussian noise with standard deviation 0.2. One of the images in the second row was obtained by corrupting the square image in the top row with "occlusive" noise: each pixel value stayed the same with probability 0.9, was replaced by 0 with probability 0.05, and was replaced by 3 with probability 0.05. Indicate which image in the second row corresponds to each type of noise noise (uniform, Gaussian, occlusive), and fully explain your answers.

b. (15 points) The three images in the third row were obtained by applying a median filter with a 3×3 window to the three images in the second row. Indicate which image in the third row corresponds to each image in the second row, and fully explain your answers.

Solution. (a) Occlusive noise must correspond to the image where about 90% of the pixels are the same as in the original image, and the remaining pixels are black and white. This is the rightmost image in the second row. A Gaussian random variable is very unlikely to take on values which are more than three standard deviations away from its mean. We would therefore expect the image corrupted by Gaussian noise with standard deviation 0.2 to have values in the range between 0.4 and 2.6, and most of the values to be fairly close to the original 1 and 2. Since uniform noise ranges from -1 to 1, we would expect many more very dark and very light pixels in the resulting image. Conclusion: the left image was obtained by adding Gaussian noise, the center image was obtained by adding uniform noise.

(b) As shown in class, median filtering is very effective in removing isolated impulses, and therefore it will clean up occlusive noise. The left image in the third row corresponds to applying the median filter to the right image of the second row. Since median filtering cannot increase local maxima (i.e. produce very bright pixels where there were none), the right image in the third row could not possibly result from applying a median filter to the left image in the second row. Therefore, the right image in the third row corresponds to the center image in the second row, and the center image in the third row corresponds to the left image in the second row.

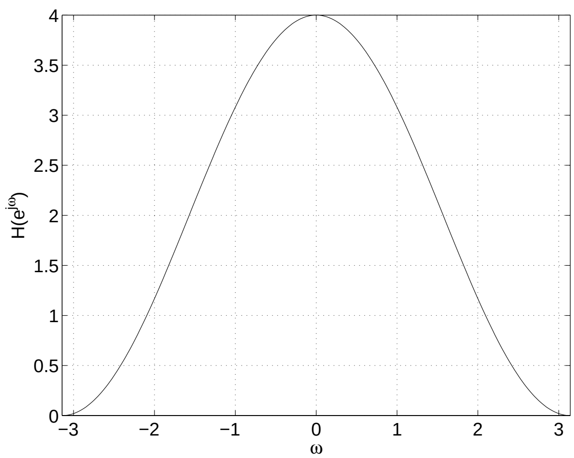


Figure 2: The frequency response plot for 5(c).

Problem 5. (20 points). Let $h(m,n)$ be the following 3×3 image:

$h(m,n) =$

0	1	0
1	4	1
0	1	0

We assume that the location of the central pixel is $(0,0)$ (i.e., $h(0,0) = 4$), and the values of h outside of this 3×3 square are zero: $h(m,n) = 0$ when $|m| \geq 2$ or $|n| \geq 2$.

Let $f(m,n) = m + n$ for all integer m and n.

a. (5 points) Calculate $f * *h(m,n)$, the 2-D convolution of $f(m,n)$ and $h(m,n)$. Give the answer for all integer m and n.

b. (5 points) Calculate $H(e^{j\omega_1}, e^{j\omega_2})$, the 2-D discrete-space Fourier transform of $h(m,n)$.

c. (5 points) Plot $H(e^{j\omega_1}, e^{j\pi})$ as a function of ω_1, for $-\pi \leq \omega_1 \leq \pi$.

d. (5 points) Calculate $g(m,n)$, the result of the following median filtering operation applied to the image $f(m,n) = m + n$:

$$g(m,n) = \text{median}\{f(m,n), f(m-1,n), f(m+1,n), f(m,n-1), f(m,n+1)\}.$$

Solution.

a. The answer is:

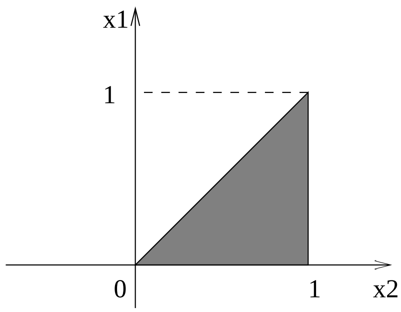
$$\begin{aligned} &4f(m,n) + f(m-1,n) + f(m+1,n) + f(m,n-1) + f(m,n+1) \\ &= 4(m+n) + (m-1+n) + (m+1+n) + (m+n-1) + (m+n+1) \\ &= 8(m+n). \end{aligned}$$

b.

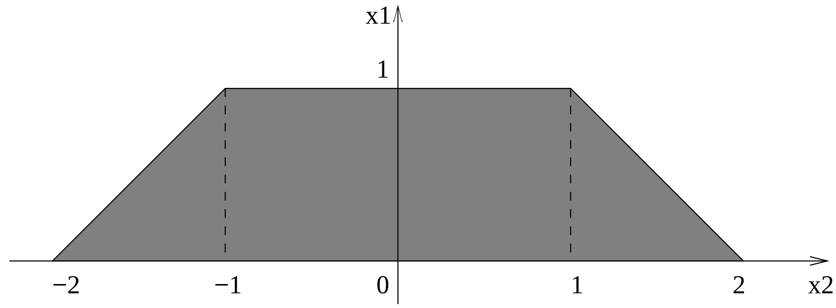
$$\begin{aligned}
H(e^{j\omega_1}, e^{j\pi}) &= h(0,0) + h(-1,0)e^{-j\omega_1} + h(1,0)e^{j\omega_1} + h(0,-1)e^{-j\omega_2} + h(0,1)e^{j\omega_2} \\
&= 4 + e^{-j\omega_1} + e^{j\omega_1} + e^{-j\omega_2} + e^{j\omega_2} \\
&= 4 + 2\cos(\omega_1) + 2\cos(\omega_2).
\end{aligned}$$

c. $H(e^{j\omega_1}, e^{j\pi}) = 4 + 2\cos(\omega_1) + 2\cos(\pi) = 2 + 2\cos(\omega_1)$, see Fig. 2.

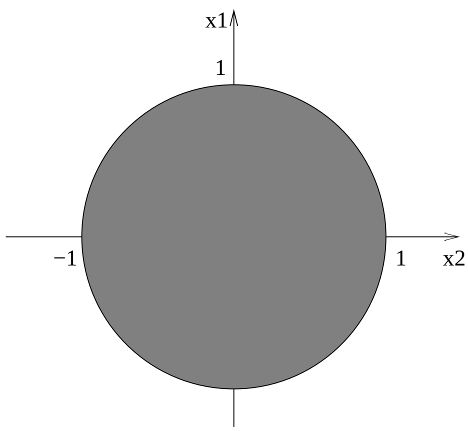
d. $g(m,n) = \text{median}\{(m+n), (m-1+n), (m+1+n), (m+n-1), (m+n+1)\} = m+n.$



(i) Support of $f_1(x_1, x_2)$.



(ii) Support of $f_2(x_1, x_2)$.



(iii) Support of $f_3(x_1, x_2)$.

Figure 3: Supports of the functions f_1, f_2, and f_3 for Problem 6.

Problem 6 (15 points). Consider three following functions of two continuous variables, x_1 and x_2:

$$f_1(x_1, x_2) = \begin{cases} x_2, & \text{inside the triangle in Fig. 3(i)} \\ 0, & \text{elsewhere} \end{cases}$$

$$f_2(x_1, x_2) = \begin{cases} 1, & \text{inside the trapezoid in Fig. 3(ii)} \\ 0, & \text{elsewhere} \end{cases}$$

$$f_3(x_1, x_2) = \begin{cases} 2, & \text{inside the circle in Fig. 3(iii)} \\ 0, & \text{elsewhere} \end{cases}$$

Among the following four functions of a continuous variable t, exactly one is a tomographic projection of f_1, exactly one is a tomographic projection of f_2, and exactly one is a tomographic projection of f_3:

$$g_a(t) = \begin{cases} 2t + 4, & -1 \le t \le 0 \\ 0, & \text{otherwise} \end{cases}$$

$$g_b(t) = \begin{cases} 1, & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$g_c(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$g_d(t) = \begin{cases} 4\sqrt{1-t^2}, & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

The angles corresponding to these tomographic projections are not given to you. Match each function f_1, f_2, and f_3 with its tomographic projection, and explain your answers.

Solution. The tomographic projection of $f_3(x_1, x_2)$ for any angle is zero for $|t| > 1$ and $4\sqrt{1-t^2}$ for $|t| \le 1$, which is $g_d(t)$. There does not exist an angle such that $g_b(t)$ is a tomographic projection for either $f_1(x_1, x_2)$ or $f_2(x_1, x_2)$. Moreover, since the support of $f_2(x_1, x_2)$ is a polygon, and since it is constant inside this polygon, its tomographic projections must be piecewise linear, which means that $g_a(t)$ is a tomographic projection for $f_2(x_1, x_2)$. (Specifically, it is the projection obtained by rotating the (x_1, x_2) coordinates clockwise by $\pi/2$.) The remaining projection $g_c(t)$ must therefore correspond to $f_1(x_1, x_2)$. (Specifically, it is the tomographic projection of $f_1(x_1, x_2)$ at the angle $\theta = 0$.)

Problem 7 (10 points). Fibonacci numbers $x_0, x_1, x_2, \ldots$ are defined as follows:

- the first two Fibonacci numbers are equal to one: $x_0 = x_1 = 1$;
- any other Fibonacci number is obtained by adding together two previous Fibonacci numbers:

$$x_{n+1} = x_n + x_{n-1} \text{ for } n \geq 1.$$

Find an explicit formula for x_n. Your formula may only consist of arithmetic operations (including taking a square root), real numbers, and the letter "n". The formula must not contain any x's.

Solution. We guess the solution of the form $x_n = a^n$ and substitute it into $x_{n+1} = x_n + x_{n-1}$:

$$\begin{aligned} a^{n+1} &= a^n + a^{n-1} \\ a^2 &= a + 1 \\ a &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

This means that the solution to the recursion is

$$x_n = \alpha \left(\frac{1+\sqrt{5}}{2} \right)^n + \beta \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Now we use this together with the initial conditions $x_0 = x_1 = 1$:

$$\begin{aligned} x_0 &= \alpha + \beta = 1 \Rightarrow \beta = 1 - \alpha \\ x_1 &= \alpha \left(\frac{1+\sqrt{5}}{2} \right) + \beta \left(\frac{1-\sqrt{5}}{2} \right) = 1 \\ &\Rightarrow \alpha = \frac{5+\sqrt{5}}{10}, \ \beta = \frac{5-\sqrt{5}}{10}. \end{aligned}$$

Therefore, the formula for the Fibonacci numbers is:

$$\frac{5+\sqrt{5}}{10} \cdot \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{5-\sqrt{5}}{10} \cdot \left(\frac{1-\sqrt{5}}{2} \right)^n.$$